The Weil representation is a representation

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Abstract

We show that the Weil representation for discriminant forms of even lattices with even signature really is a representation of $\mathrm{SL}_2(\mathbb{Z})$ on the group ring $\mathbb{C}[D]$. For a much more detailed treatment of discriminant forms, see [WeMSc] and the references therein.

We want to define a representation on the group $\mathrm{SL}_2(\mathbb{Z})$ which is defined as follows:

1 Definition.

$$SL_2(\mathbb{Z}) := \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2\times 2} \;\middle|\; \det(M) = ad - bc = 1 \right\}$$

2 Definition. *Let $A = \{a_1, ..., a_{|A|}\}$ be a finite group and let K be a field. We define the $|A|$-dimensional K vector space $K[A]$ to be $K^{|A|}$. In $K[A]$ we rename the standard basis $e_1, ..., e_{|A|}$ to $\mathbf{e}_{a_1}, ..., \mathbf{e}_{a_{|A|}}$ and we declare a multiplication on $K[A]$ by*

$$\mathbf{e}_a \cdot \mathbf{e}_b := \mathbf{e}_{a \cdot b}$$

therefore turning $K[A]$ into a K-algebra, the so-called group ring of A.

3 Definition. *Let G be a group and V be a K-vector space for some field K. Let $\mathrm{Aut}_K(V)$ denote all K-linear bijective maps from V to V. A representation (of G on V) is a map $\rho : G \mapsto \mathrm{Aut}_K(V)$ that is itself a homomorphism of groups.*

In the case of vector spaces one obtains homomorphisms by just dictating what the map should do on a basis. In the case of groups it is not clear what a basis should mean. We will later on define a representation on a certain group by just dictating what the map should do on the so-called generators (for more information on generators and free groups see either [Al2], [JS] or any other book on abstract algebra).

4 Theorem. *There is an isomorphism*

$$\eta : \langle s, u \mid s^2 = u^3, s^4 = id \rangle \mapsto SL_2(\mathbb{Z})$$

satisfying $\eta(s) = S$ and $\eta(u) = TS =: U$. Thus, $SL_2(\mathbb{Z})$ is the free group generated by S, U modulo the relations $s^4, u^3 s^{-2}$ where $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Proof. See [Al2]. □

We now define the Weil representation, a representation of $\mathrm{SL}_2(\mathbb{Z})$ on the $\mathbb{C}$-vector space $\mathbb{C}[D]$:

5 Definition. *Let $(L, \langle\cdot,\cdot\rangle)$ be an even lattice with discriminant form D. Let $(\cdot,\cdot)$ denote the finite bilinear form on D and let $\mathbb{C}[D]$ be the group ring. For $\gamma \in D$ we define*

$$\begin{aligned} \rho(T).\mathbf{e}_\gamma &= e\left(-Q(\gamma)\right)\mathbf{e}_\gamma \\ \rho(S).\mathbf{e}_\gamma &= \frac{e\left(\mathrm{sig}(D)/8\right)}{\sqrt{|D|}} \sum_{\beta\in D} e\left((\gamma,\beta)\right)\mathbf{e}_\beta \end{aligned}$$

We also write $T.v$ and $S.v$ in place of $\rho(T).v$ and $\rho(S).v$.

6 Theorem. *Let $(L, \langle\cdot,\cdot\rangle)$ be an even lattice, D its discriminant form with even signature $\mathrm{sig}(D)$, then $\mathbb{C}[D] \mapsto \mathbb{C}[D] : v \mapsto T.v$ and $\mathbb{C}[D] \mapsto \mathbb{C}[D] : v \mapsto S.v$ are K-linear vector space automorphisms on $\mathbb{C}[D]$. Setting $\rho(T^{-1}).\mathbf{e}_\gamma := \rho(T)^{-1}.\mathbf{e}_\gamma$ and $\rho(S^{-1}).\mathbf{e}_\gamma := \rho(S)^{-1}.\mathbf{e}_\gamma$ yields a representation of $SL_2(\mathbb{Z})$ on the free group generated by S, T. Further,*

$$\begin{aligned} \rho((TS)^3).\mathbf{e}_\gamma &= e(\mathrm{sig}(D)/8)\mathbf{e}_{-\gamma} = \rho(S^2).\mathbf{e}_\gamma \\ \rho(S^4).\mathbf{e}_\gamma &= \mathbf{e}_\gamma \end{aligned}$$

so that ρ extends to a representation of $\langle s, u \mid s^2 = id, s^2 = u^3\rangle$ and since this group is isomorphic to $SL_2(\mathbb{Z})$, ρ defines a representation on $SL_2(\mathbb{Z})$.

Before we start the proof we need a lemma:

7 Lemma. *Let D be as described above. Then*

(a) Milgrams formula:

$$\sum_{\gamma\in D} e(Q(\gamma)) = \sqrt{|D|}e\left(\frac{\mathrm{sig}(D)}{8}\right)$$

(b) For any $\alpha \in D$, we have

$$\sum_{\gamma\in D} e(-Q(\gamma) + (\alpha,\gamma)) = e(Q(\alpha))\sqrt{|D|}e\left(-\frac{\mathrm{sig}(D)}{8}\right)$$

(c) For any $\alpha \in D$, we have

$$\sum_{\gamma \in D} e(\alpha, \gamma) = \begin{cases} |D| & \text{if } \alpha = 0 + L \\ 0 & \text{otherwise} \end{cases}$$

Proof. (a): See [Mil], Appendix 4, the first theorem. (b): Since

$$\begin{aligned} Q(\gamma + (-\alpha)) - Q(\gamma) - Q((-\alpha)) &= (\gamma, -\alpha) \\ \implies Q(\gamma) &= Q(\gamma - \alpha) - Q(\alpha) + (\gamma, \alpha) \end{aligned}$$

we have

$$\begin{aligned} \sum_{\gamma \in D} e(-Q(\gamma) + (\alpha, \gamma)) &= \sum_{\gamma \in D} e(-Q(\gamma - \alpha) + Q(\alpha) - (\gamma, \alpha) + (\alpha, \gamma)) \\ &= e(Q(\alpha)) \sum_{\gamma \in D} e(-Q(\gamma - \alpha)) \\ &= e(Q(\alpha)) \sum_{\gamma \in D} e(-Q(\gamma)) \qquad (0.1) \\ &\quad \text{(as } \gamma \mapsto \gamma - \alpha \text{ is a bijection on } D\text{)} \end{aligned}$$

Taking the complex conjugate of Milgram's formula yields

$$\sum_{\gamma \in D} e\left(-Q(\gamma)\right) = \overline{\sum_{\gamma \in D} e(Q(\gamma))} = \overline{\sqrt{|D|} e\left(\frac{\operatorname{sig}(D)}{8}\right)} = \sqrt{|D|} e\left(\frac{-\operatorname{sig}(D)}{8}\right)$$

so that

$$\sum_{\gamma \in D} e(-Q(\gamma) + (\alpha, \gamma)) \overset{(0.1)}{=} e(Q(\alpha)) \sum_{\gamma \in D} e(-Q(\gamma)) = e(Q(\alpha)) \sqrt{D} e(-\operatorname{sig}(D)/8)$$

(c) Consider the mapping $\chi : D \mapsto \mathbb{C}^\times, \chi(\gamma) := e(\gamma, \alpha)$ then this map is a character of D. It is trivial if and only if $\alpha = 0 + L$ where the only if part comes from the fact that $(\cdot, \cdot)$ is non-degenerate (i.e. there is some $\gamma \in D$ such that $(\gamma, \alpha) \neq 0 + \mathbb{Z}$ and hence $\chi(\gamma) \neq 1$). Consequently, if $\alpha \neq 0 + L$

$$\sum_{\gamma \in D} e(\alpha, \gamma) = \sum_{\gamma \in D} \chi(\gamma) = 0$$

by basic algebra (use the geometric sum!). □

Proof of Thm 6. Concerning the bijectivity of $\rho(S)$ and the relation $S^4 = Id$: Set $c_D := \frac{e(\text{sig}(D)/8)}{\sqrt{|D|}}$, then

$$\begin{aligned}
\rho(S)\rho(S).\mathbf{e}_\gamma &= c_D^2 \sum_{\beta\in D} e(\beta,\gamma) \sum_{\lambda\in D} e(\beta,\lambda)\mathbf{e}_\lambda \\
&= c_D^2 \sum_{\lambda\in D} \underbrace{\sum_{\beta\in D} e(\beta,\gamma+\lambda)}_{=0 \text{ except } \gamma+\lambda=0 \iff \lambda=-\gamma \text{ by 7 (c)}} \mathbf{e}_\lambda \\
&= \frac{e(\text{sig}(D)/4)}{\cancel{|D|}} \cancel{|D|} \mathbf{e}_{-\gamma}
\end{aligned}$$

In particular, $\rho(S)^4.\mathbf{e}_\gamma = e(\text{sig}(D)/2)\mathbf{e}_{--\gamma} = \mathbf{e}_\gamma$ because the signature of L is even by assumption. Therefore, $\rho(S)\rho(S)^3 = \rho(S)^3\rho(S) = Id$ on $\mathbb{C}[D]$, i.e. $\rho(S)$ is invertible and the relation $S^4 = Id$ is respected by ρ.

The bijectivity of $\rho(T)$ is clear. On the relation $U^3 = S^2$: We have

$$\rho(U).\mathbf{e}_* = \rho(T)\rho(S)\mathbf{e}_* = c_D \sum_{\beta\in D} e(\beta,*)\rho(T)\mathbf{e}_\beta = c_D \sum_{\beta\in D} e(-Q(\beta)+(\beta,*))\mathbf{e}_\beta$$

Set $b_D := \sqrt{|D|}e\left(\frac{-\text{sig}(D)}{8}\right)$ then

$$
\begin{aligned}
\rho(ST)^3.\mathbf{e}_\gamma &= \rho(ST) c_D \sum_{\beta\in D} e(-Q(\beta)+(\beta,\gamma))\rho(ST)\mathbf{e}_\beta \\
&= c_D^2 \sum_{\beta\in D} e(-Q(\beta)+(\beta,\gamma)) \sum_{\lambda\in D} e(-Q(\lambda)+(\lambda,\beta))\rho(ST)\mathbf{e}_\lambda \\
&= c_D^3 \sum_{\beta\in D} e(-Q(\beta)+(\beta,\gamma)) \sum_{\lambda\in D} e(-Q(\lambda)+(\lambda,\beta)) \sum_{\delta\in D} e(-Q(\delta)+(\delta,\lambda))\mathbf{e}_\delta \\
&= c_D^3 \sum_{\delta\in D} e(-Q(\delta)) \sum_{\beta\in D} e(-Q(\beta)+(\beta,\gamma)) \underbrace{\sum_{\lambda\in D} e(-Q(\lambda)+(\lambda,\beta+\delta))}_{=b_D e(Q(\beta+\delta))(\text{by } 7(\text{b}))} \mathbf{e}_\delta \\
&= c_D^3 b_D \sum_{\delta\in D} \cancel{e(-Q(\delta))} \sum_{\beta\in D} e(\cancel{-Q(\beta)}+(\beta,\gamma)) \quad e(\cancel{Q(\beta)}+\cancel{Q(\delta)}+(\beta,\delta))\mathbf{e}_\delta \\
&= c_D^3 b_D \sum_{\delta\in D} \qquad \underbrace{\sum_{\beta\in D} e((\beta,\gamma+\delta))}_{=0 \text{ if } \delta\neq-\gamma \text{ and } =|D| \text{ otherwise, see } 7(\text{c})} \qquad \mathbf{e}_\delta \\
&= c_D^3 b_D |D| \mathbf{e}_{-\gamma}
\end{aligned}
$$

Since

$$
c_D^3 b_D |D| = \left(\frac{e(\mathrm{sig}(D)/8)}{\sqrt{|D|}}\right)^3 e(-\mathrm{sig}(D)/8)\sqrt{|D|}\sqrt{|D|}^2 = e(\mathrm{sig}(D)/4)
$$

we have shown that

$$
\rho(TS)^3.\mathbf{e}_\gamma = e(\mathrm{sig}(D)/4)\mathbf{e}_{-\gamma} = \rho(S)^2.\mathbf{e}_\gamma
$$

so that ρ is compatible with the relation $u^3 = s^2$ too and $\rho(U)\rho(U)^5 = \rho(U)^5\rho(U) = Id$, i.e. $\rho(U)$ is invertible. □

References

[JS] Jantzen, Schwermer: *Algebra.*

[WeMSc] Fabian Werner, Hecke Operators and the Weil representation. Masters thesis, 2011, `http://happy-werner.de/uni/Mathe_MSc/mthesis.pdf`.

[Mil] John Milnor, SYMMETRIC BILINEAR FORMS, Springer, 1973.

[Al2] $\mathrm{SL}_2(\mathbb{Z})$ IS THE FREE GROUP $\langle s, u \mid s^4 = id, s^2 = u^3 \rangle$ `http://happy-werner.de/uni/sonstiges/sl2Z.pdf`