Bachelor Thesis

Rings of modular forms

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Erklärung an Eides Statt

Hiermit versichere ich, dass ich die vorliegende Arbeit ohne Hilfe Dritter und nur mit den angegebenen Quellen und Hilfsmitteln angefertigt habe. Ich habe alle Stellen, die ich aus den Quellen wörtlich oder inhaltlich entnommen habe, als solche kenntlich gemacht. Diese Arbeit hat in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegen.

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1 Introduction

The ultimate goal in this bachelor thesis is to show that the ring of modular forms with *integer* weight for two specific subgroups of $\mathrm{SL}_2(\mathbb{Z})$, namely $\Gamma_0(2)$ and $\Gamma_1(3)$, is generated by two fixed modular forms. In order to do so, we need three main ingredients:

① Of course, the two basic modular forms for the subgroups announced need to exist.

② We need an assertion of the form 'There are at most x essentially different modular forms of weight k for the subgroup'.

③ We need to assure that we can construct these x essentially different modular forms of weight k.

The term 'essentially different' will turn out to mean 'linearly independent' as the set of modular forms is a vector space.

① will only be mentioned briefly but references will be given where to find the full proofs for the constructions. ② will be done using the so-called $k/12$ – Formula which will give an upper bound on the dimension of the space of modular forms of a specific weight. On ③: We will construct all modular forms as monomials in the two basic modular forms. In order for the monomials to be linearly independent, we need the two basic modular forms to be algebraically independent. This can be shown if one has enough information about the zeros of the modular forms. The desire for this will lead to the concept of the "order" of a modular form f at some point z in $\mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$. The order will essentially be $\omega_f(z)$ (the index of the first occurring term in the Laurent series of f developed around z) up to multiplication of a number $c_z \in \{1, 1/2, 1/3\}$ depending on the point z.

In the next section, we are going to precisely define the terms "$\mathrm{SL}_2(\mathbb{Z})$ ", "modular form" and so on. In section 3 the $k/12$ – Formula for $\mathrm{SL}_2(\mathbb{Z})$ and for general subgroups will be deduced. Finally, in section 4, we prove the rings of modular forms for $\Gamma_0(2)$ and $\Gamma_1(3)$ to be polynomial rings in two variables.

Summarized the following results have been established:

- The general $k/12$ – Formula has been extended to the case where the subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ does not contain $-Id$ where in [Ra 77] you will only find the proof for the case where $-Id \in \Gamma$.

- We show that there is only one fixed point orbit for the subgroups $\Gamma_0(2), \Gamma_1(3)$. The proof for this was already done in [Sko 92] but we will prove it without the usage of the Hurwitz summation formula or any Riemann surface theory. We will only use elementary computations.
- We formally prove the claim from [B 00] (see p. 27, 5th sentence) that the ring of modular forms of $\Gamma_1(3)$ of integer weight also forms a polynomial ring in two variables.

Throughout this document we will follow the notation from [Ra 77].

2 Basic concepts and definitions

2.1 Actions of the modular group and its subgroups

2.1.1 Definition. *We define*

$$SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{Z}^{2x2} \;\middle|\; \det(T) = ad - bc = 1 \right\}$$

$\mathrm{SL}_2(\mathbb{Z})$ is a group. The inverse of $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is given by $\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$. In n dimensions this generally follows from Cramer's rule.

2.1.2 Definition. *We define the subgroups*

(a) $\Gamma_0(2) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; c \equiv 0 \mod N \right\}$

(b) $\Gamma_1(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod N \right\}$

Where the asterisk means that the value is arbitrary, i.e. $n \equiv * \mod N$ *is always true for all* $n \in \mathbb{Z}$.

That these sets are subgroups is shown by an elementary calculation.

2.1.3 Definition. *Let* Γ *be an arbitrary subgroup of* $SL_2(\mathbb{Z})$.

(a) For matrices $S, T \in SL_2(\mathbb{Z})$ *we define the equivalence relation* $T \sim_\Gamma S \iff (\exists G \in \Gamma)\ T = G \cdot S$ *and* $[T]_\Gamma := \{S \in SL_2(\mathbb{Z}) \mid S \sim_\Gamma T\}$ *its equivalence class. We will in short just write* $[T]$ *instead of* $[T]_\Gamma$ *and* $\sim$ *instead of* $\sim_\Gamma$ *as it will always be clear from the context which particular subgroup* Γ *will be meant.*

(b) *The* index *of* Γ *is* $[SL_2(\mathbb{Z}):\Gamma] := |SL_2(\mathbb{Z})/\Gamma| := |SL_2(\mathbb{Z})/\sim|$ *where* $SL_2(\mathbb{Z})/\sim$ *denotes the quotient of set modulo equivalency relation, i.e.* $SL_2(\mathbb{Z})/\sim \; := \{[T] \mid T \in SL_2(\mathbb{Z})\}$. *It is possible that* $[SL_2(\mathbb{Z}):\Gamma]$ *is infinite.*

(c) *A set* $\mathcal{R} \subseteq SL_2(\mathbb{Z})$ *is called a* system of right representatives *(RRS for short) if it generates* $SL_2(\mathbb{Z})/\sim$ *i.e. if* $\{[R] \mid R \in \mathcal{R}\} = SL_2(\mathbb{Z})/\sim$ *and* $R \nsim R'$ *whenever* $R \neq R'$ *for all* $R, R' \in R$. *In this case we have* $[SL_2(\mathbb{Z}):\Gamma] = |\mathcal{R}|$ *as* $R \mapsto [R]$ *is a bijection from* $\mathcal{R}$ *to* $SL_2(\mathbb{Z})/\sim$.

Given any subgroup H of a group G, we can always select an RRS by the Axiom of choice, by the Lemma of Zorn respectively if we sort the sets that contain only Elements x, y s.t. $x \nsim y$ with the partial "$\subset$"-relation. Then, every chain possesses a maximal element, namely the union over all its members. Therefore there must be a maximal element for the "$\subset$"-relation called $\mathcal{M}$. This must be an RRS as all elements are inequivalent and if there is some class of which no representative is in $\mathcal{M}$, then we simply add a representative of this new class to $\mathcal{M}$ and obtain a bigger set which is a contradiction. Hence we have shown:

2.1.4 Lemma. *For every subgroup* Γ *of* $SL_2(\mathbb{Z})$*, there exists an RRS.*

Although it is possible that $[\mathrm{SL}_2(\mathbb{Z}):\Gamma] = \infty$, we will always assume that this is not the case. This is reasonable, because the subgroups that we will be concerned about ($\Gamma_0(2)$ and $\Gamma_1(3)$) do indeed have a finite index in $\mathrm{SL}_2(\mathbb{Z})$.

We will now define the action that we will be mainly concerned about. First of all we compactify $\mathbb{C}$ with the point ∞ using the Alexandroff one-point compactification (i.e. a set $O \subseteq \mathbb{C}$ is called open iff. it is already open as a subset of $\mathbb{C}$ in the usual topology or O^C is compact in $\mathbb{C}$). We then define

$$
\begin{aligned}
\overline{\mathbb{C}} &:= \mathbb{C} \cup \{\infty\} \\
\mathbb{H} &:= \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\} \\
\mathbb{P} &:= \mathbb{Q} \cup \{\infty\} \\
\overline{\mathbb{H}} &:= \mathbb{H} \cup \mathbb{P}
\end{aligned}
$$

We also enrich the multiplicative and additive structure on $\mathbb{C}$ by the definitions

$$\infty + z = z + \infty = \infty, \quad \frac{z}{\infty} = 0 \;\; \forall z \in \mathbb{C}$$

$$z \cdot \infty = \infty \cdot z = \infty, \quad \frac{z}{0} = \infty \quad \forall z \in \overline{\mathbb{C}} \setminus \{0\}$$

Note that there is no rule for $0 \cdot \infty$ and other special cases.

2.1.5 Definition. *On $SL_2(\mathbb{Z}) \times \overline{\mathbb{C}}$ we define the operation "." for $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \overline{\mathbb{C}}$ to be*

$$T.z := \frac{az+b}{cz+d} \quad (z \in \mathbb{C}), \quad T.\infty := \frac{a}{c} \quad \textit{and} \quad T.\left(\frac{-d}{c}\right) := \infty$$

.

An elementary computation shows that this indeed defines a group action. For this reason, when given $S, T \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \overline{\mathbb{C}}$, we will write STz for $(S \cdot T).z = S.(T.z)$ and Tz for $T.z$. We now show that "." can be restricted to a group operation on $\overline{\mathbb{H}}$ as this is the operation we will be interested in. Let $T \in \mathrm{SL}_2(\mathbb{Z})$, then $T.\mathbb{P} \subseteq \mathbb{P}$ by definition. If $z = x+iy \in \mathbb{H}$ is given, then $T.z \in \mathbb{H}$ because for $z = x + iy$:

$$\begin{aligned}
\mathrm{Im}(Tz) &= \mathrm{Im}\left(\frac{az+b}{cz+d}\right) \\
&= \mathrm{Im}\left((az+b)\frac{1}{(cx+d)+i(cy)}\right) \\
&= \mathrm{Im}\left([a(x+iy)+b] \cdot [(cx+d)-i(cy)] \cdot \frac{1}{|cz+d|^2}\right) \\
&= \mathrm{Im}\Big([((ax+b)(cx+d)+(ay)(cy)) \\
&\qquad + i((ay)(cx+d)-(ax+b)(cy))] \cdot \frac{1}{|cz+d|^2}\Big) \\
&= [(ay)(cx+d)-(ax+b)(cy)]\frac{1}{|cz+d|^2} \\
&= [aycx+ady-axcy-bcy]\frac{1}{|cz+d|^2} \\
&= \det(T)\underbrace{\frac{1}{|cz+d|^2}}_{>0} \\
&> 0
\end{aligned}$$

as $cz+d \neq 0$ must hold because otherwise $z = -d/c \notin \mathbb{H}$. So all $T \in \mathrm{SL}_2(\mathbb{Z})$ can be regarded as bijections from $\overline{\mathbb{H}}$ to $\overline{\mathbb{H}}$ as the inverse mapping is given by the inverse of the Matrix which again maps from $\overline{\mathbb{H}}$ to $\overline{\mathbb{H}}$ by the same arguments as above.

If $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ then we also define $-T := \left(\begin{smallmatrix} -a & -b \\ -c & -d \end{smallmatrix}\right)$. An important observation is that $-T$ and T are in general unequal but both define the same action as $Tz = \frac{az+b}{cz+d} = \frac{(-1)((-a)z+(-b))}{(-1)((-c)z+(-d))} = \frac{(-a)z+(-b)}{(-c)z+(-d)} = (-T)z$. One may wonder whether this happens more often. This question will be answered now:

2.1.6 Definition. *We define the map*

$$\Phi : SL_2(\mathbb{Z}) \longmapsto \{\textit{invertible mappings from } \overline{\mathbb{H}} \textit{ to } \overline{\mathbb{H}}\}$$

as $\Phi(T) = \widehat{T}$ *where* $\widehat{T} : \overline{\mathbb{H}} \to \overline{\mathbb{H}}, \widehat{T}(z) = T.z$. *For any subgroup* Γ *of* $SL_2(\mathbb{Z})$ *we also define* $\widehat{\Gamma} := \Phi(\Gamma)$ *and call* $\widehat{\Gamma}$ *the* homogeneous group *associated to* Γ.

Because "." is a group operation, Φ is multiplicative in the sense that $\Phi(ST) = \Phi(S) \circ \Phi(T) =: \Phi(S)\Phi(T)$. As an abbreviation we will also write $\widehat{T}$ for $\Phi(T)$.

2.1.7 Lemma. *Let* Γ *be a subgroup of* $SL_2(\mathbb{Z})$ *and* $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, *then*

(a) $c = 0 \Rightarrow a = d = 1$ *or* $a = d = -1$

(b) There are z_1, z_2, z_3 *all pairwise unequal such that* $z_i \neq -\frac{d}{c}, z_i \neq \infty$ *and* $Tz_i = z_i$ *for* $i = 1, 2, 3$ *then* $T = +Id$ *or* $T = -Id$.

(c) The kernel of Φ *is* $\{+Id, -Id\}$.

(d) If $-Id \in \Gamma$, *then* $\widehat{\Gamma} = \text{Image}(\Phi) \cong \Gamma / \ker(\Phi) = \Gamma / \{\pm Id\}$.

Proof. (a) $c = 0$ implies $1 = \det(T) = ad - cb = ad$ i.e. $a, d \in \mathbb{Z}$ and multiplicatively invertible so both must be ± 1. As they must produce $+1$ when multiplicated, the cases with different signs are impossible.

(b) We reformulate $Tz = z$:

$$\begin{aligned} Tz = z &\iff \frac{az+b}{cz+d} = z \\ &\iff az + b = cz^2 + dz (\text{as} z \neq \infty, cz + d \neq 0) \\ &\iff cz^2 + (d-a)z - b = 0 \end{aligned}$$

Now this polynomial is of degree 2 but has three different roots. Consequently it has to be the zero polynomial, i.e. $c = 0 = b$ and by (a), $a = d = \pm 1$.

(c) $\Phi(T) = \widehat{Id}$ then $\widehat{T}.z = \widehat{Id}.z = z$ for all $z \in \overline{\mathbb{H}}$ then in particular for $i, 2i, 3i \in \overline{\mathbb{H}} \setminus \{\infty, -d/c\}$. Now apply (b).

(d) Application of the first isomorphism theorem from general algebra. $\square$

We will often be confronted with equations of the form $\widehat{S} = \widehat{T}$. As a consequence of Lemma 2.1.7(c), $S = +T$ or $S = -T$ follows. For brevity we will from now on write $\pm T$ for the fact that the current assertion holds either for $+T$ or for $-T$, so for example $\ker(\Phi) = \{\pm Id\}$. We also clearly want to distinguish between assertions in $\Phi(\mathrm{SL}_2(\mathbb{Z}))$ and $\mathrm{SL}_2(\mathbb{Z})$. With the "hat"-notation we want to emphasis that we now talk about actions (i.e. mappings) induced by matrices and with the $\pm$-symbol we want to emphasis that we now work with actual matrices (not mappings!) in $\mathrm{SL}_2(\mathbb{Z})$.

2.1.8 Lemma. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having a finite index $n = [\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma}]$, then*

(a)

$$[SL_2(\mathbb{Z}) : \Gamma] = \begin{cases} [\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma}] & \text{if } \ -Id \in \Gamma \\ 2[\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma}] & \text{otherwise} \end{cases}$$

In particular, if $-Id \notin \Gamma$, then $[SL_2(\mathbb{Z}) : \Gamma]$ is even.

(b) If $\widehat{\mathcal{R}} = \{\widehat{L_1}, ..., \widehat{L_n}\}$ is an RRS of $\widehat{SL_2(\mathbb{Z})}/\widehat{\Gamma}$, then

$$\mathcal{R} := \begin{cases} \{L_1, ..., L_n\} & \text{if } \ -Id \in \Gamma \\ \{+L_1, -L_1, ..., +L_n, -L_n\} & \text{otherwise} \end{cases}$$

is an RRS of $SL_2(\mathbb{Z})/\Gamma$.

(c) If conversely $k = [SL_2(\mathbb{Z}) : \Gamma]$ and $\mathcal{R} = \{L_1, ..., L_k\}$ is an RRS for $SL_2(\mathbb{Z})/\Gamma$ then if $-Id \in \Gamma$, $\widehat{\mathcal{R}} := \{\widehat{L_1}, ..., \widehat{L_k}\}$ is an RRS for $\widehat{SL_2(\mathbb{Z})}/\widehat{\Gamma}$. If $-Id \notin \Gamma$, then precisely half of the elements in $\widehat{\mathcal{R}}$ pair up in the sense that after resorting the $\widehat{L_i}$, we have

$$\Phi(L_1) \sim \Phi(L_{(n/2)+1}),\ \Phi(L_2) \sim \Phi(L_{(n/2)+2}), ..., \Phi(L_{n/2}) \sim \Phi(L_n)$$

and $\{\Phi(L_1), ..., \Phi(L_{n/2})\}$ forms an RRS for $\widehat{SL_2(\mathbb{Z})}/\Gamma$.

Proof. (a): follows from (b) and the fact that the size of the RRS determines the index.

(b): Let $n = [\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}]$ and $\widehat{\mathcal{R}} = \{\widehat{R_1}, ..., \widehat{R_n}\}$ (i.e. $R_1, ..., R_n \in \mathrm{SL}_2(\mathbb{Z})$ and $\widehat{R_i} = \Phi(R_i)$) be a RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ over $\widehat{\Gamma}$. At first we consider the case where $-Id \in \Gamma$. Consider the set $\mathcal{R} := \{R_1, ..., R_n\}$ then we claim that $\mathcal{R}$ is an RRS for $\mathrm{SL}_2(\mathbb{Z})$ over Γ. The $R \in \mathcal{R}$ are incongruent for if $R_i \sim R_j$ then there is a $G \in \Gamma$ with $R_i = GR_j$ so that $\Phi(R_i) = \Phi(G)\Phi(R_j)$ which is impossible as $\widehat{\mathcal{R}}$ was an RRS. It generates $\mathrm{SL}_2(\mathbb{Z})$: Let some matrix $T \in \mathrm{SL}_2(\mathbb{Z})$ be given, then since $\widehat{\mathcal{R}}$ was an RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}$, there is an $i \in \{1, ..., n\}$ and a $G \in \Gamma$ such that

$$\Phi(T) = \Phi(G) \cdot \Phi(R_i) = \Phi(GR_i)$$

so that either

$$T = +GR_i \quad \text{or} \quad T = -GR_i$$

In the first case, $T \in \Gamma \cdot \mathcal{R}$ and we are done. In the latter one we set $\widetilde{G} := -Id \cdot G \in \Gamma$ so that $T = \widetilde{G}R_i \in \Gamma \cdot \mathcal{R}$.

In the case where $-Id \notin \Gamma$ we set $L_1 = R_1, L_2 = -R_1, L_3 = R_2, L_4 = -R_2, ...$ and $\mathcal{R} = \{L_1, ..., L_{2n}\} = \{+R_1, -R_1, +R_2, -R_2, ...\}$ and claim that this is an RRS. Incongruity: Assume $L_n \sim L_m$ so that there are i, j with $\epsilon R_i \sim \delta R_j,\ \epsilon, \delta \in \{+1, -1\}$. $i \neq j$ leads to

$$\Phi(R_i) = \Phi(\epsilon R_i) \in \Phi(\Gamma \delta R_j) = \Phi(\Gamma)\Phi(R_j)$$

which is a contradiction as $\widehat{\mathcal{R}}$ was an RRS. In the case $R_i \sim -R_i$ there is a $G \in \Gamma$ with $R_i = G(-R_i)$. Canceling out R_i leads to $G = -Id$ which is impossible as $-Id \notin \Gamma$ by assumption. That $\mathcal{R}$ generates $\mathrm{SL}_2(\mathbb{Z})$ is shown as above where in the case $T = -GR_i$ we cannot take $-G \in \Gamma$ but we can say that $T = G(-R_i) \in \Gamma \cdot \mathcal{R}$ as $-R_i \in \mathcal{R}$.

(c): The case $-Id \in \Gamma$ is shown analogously to the above. When $-Id \notin \Gamma$, we create the RRS announced step by step. Firstly, we know that $\widehat{\mathcal{R}}$ still generates $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ as

$$\mathrm{SL}_2(\mathbb{Z}) = \Gamma\mathcal{R} \Rightarrow \widehat{\mathrm{SL}_2(\mathbb{Z})} = \Phi(\mathrm{SL}_2(\mathbb{Z})) = \Phi(\Gamma\mathcal{R}) = \Phi(G)\Phi(\mathcal{R})$$

It cannot be the case that all $\Phi(R_i)$ are incongruent as otherwise $\Phi(\mathcal{R})$ was an RRS so $[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \Gamma] = |\Phi(\mathcal{R})| = n \overset{(a)}{=} 2[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \Gamma]$ which is a contradiction. Consequently (after resorting the $\Phi(R_i)$) $\Phi(R_1) \sim \Phi(R_{(n/2)+1})$ and thus we can set $\widehat{\mathcal{R}}^{(1)} := \widehat{\mathcal{R}} \setminus \{\Phi(R_{(n/2)+1})\}$ and still, $\widehat{\mathcal{R}}^{(1)}$ generates $\widehat{\mathrm{SL}_2(\mathbb{Z})}$. Continuing in this way yields sets $\widehat{\mathcal{R}}^{(1)}, \widehat{\mathcal{R}}^{(2)}, ...$ Note that from step to step, only different pairs of $\Phi(R_i), \Phi(R_j)$ that are equivalent can be involved. We only show this for step 1, the rest works analogously: in step 1, $\Phi(R_1) \sim \Phi(R_{(n/2)+1})$. In the next step, it cannot be $\Phi(R_1)$ that is equivalent to some other $\Phi(R_j)$ for some $j \neq (n/2)+1$ as otherwise

$$\begin{aligned}
&\Phi(R_{(n/2)+1}) \sim \Phi(R_1) \sim \Phi(R_j) \\
\Rightarrow &\exists G, H \in \Gamma \quad \widehat{G}\widehat{R_{(n/2)+1}} = \widehat{R_1} \text{ and } \widehat{H}\widehat{R_j} = \widehat{R_1} \\
\Rightarrow &GR_{(n/2)+1} = R_1 \text{ or } -GR_{(n/2)+1} = R_1 \\
&\text{and} \\
&HR_j = R_1 \text{ or } -HR_j = R_1
\end{aligned}$$

The first cases cannot occur as $\mathcal{R}$ was an RRS for $\mathrm{SL}_2(\mathbb{Z})/\Gamma$. So both times, the second case must be true:

$$\begin{aligned}
&\Rightarrow -GR_{(n/2)+1} = R_1 \text{ and } -HR_j = R_1 \\
&\Rightarrow -GR_{(n/2)+1} = -HR_j \\
&\Rightarrow GR_{(n/2)+1} = HR_j
\end{aligned}$$

by multiplying the whole equation with $-Id$. This is the same contradiction as above. All the sets $\widehat{\mathcal{R}}^{(1)}, \widehat{\mathcal{R}}^{(2)}, \ldots$ still generate $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ as this property is preserved from step to step. Thus, $\widehat{\mathcal{R}}^{(i)}$ is an RRS if all members of the set are incongruent. This must be the case for $\widehat{\mathcal{R}}^{(n/2)}$ as otherwise the size of $\widehat{\mathcal{R}}^{(i)}$ determined the index which would then be too small or too big according to (a).

$\square$

ATTENTION! As we have seen in the lemma above: it is of crucial importance whether we mean the homogenized versions ($\widehat{=}$ actions) induced by the groups (i.e. $\widehat{\Gamma}$, $\widehat{\mathrm{SL}_2(\mathbb{Z})}$ respectively) or the *actual* group as a set of matrices (i.e. Γ and $\mathrm{SL}_2(\mathbb{Z})$).

2.1.9 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$. On $\overline{\mathbb{H}}$ we define the equivalence relation $z \approx_\Gamma z' \iff \exists G \in \Gamma \; z = G.z'$ and the equivalence class $[z]_\Gamma := \{z' \in \overline{\mathbb{H}} \mid z' \approx_\Gamma z\}$, the orbit of z under the action of Γ. Again we want to suppress the relation to the concrete subgroup as it will be clear which one is meant from the context. Therefore we always write $[z] = SL_2(\mathbb{Z}).z$ for the orbit of z under $SL_2(\mathbb{Z})$ and $[\![z]\!] = \Gamma.z$ for the orbit of z under the action of the subgroup Γ. If $x \approx y$ we also say that x is congruent to y modulo Γ.*

2.1.10 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$, then a set of points $\mathbb{F}^* \subseteq \overline{\mathbb{H}}$ is called a* proper fundamental region *iff. for every $z \in \overline{\mathbb{H}}$, $|\,[\![z]\!] \cap$*

$\mathbb{F}^*| = 1$. *Equivalently one could say that all elements in* $\mathbb{F}^*$ *are inequivalent and* $\{[\![z]\!] \mid z \in \mathbb{F}^*\} = \overline{\mathbb{H}}/\approx$, *i.e. what an RRS means to* $SL_2(\mathbb{Z})/\Gamma$, $\mathbb{F}^*$ *means to* $\overline{\mathbb{H}}/\approx$.

Analogously to the case of an RRS, we can extend a given set of incongruent points (or the empty set) to a proper fundamental domain (see Lemma 2.1.4).

2.1.11 Theorem. *Set* $\mathbb{F}_1 := \{z \in \mathbb{C} \mid -\frac{1}{2} \leq \mathrm{Re}(z) \leq 0$ *and* $|z| \geq 1\}, \mathbb{F}_2 := \{z \in \mathbb{C} \mid 0 < \mathrm{Re}(z) < \frac{1}{2}$ *and* $|z| > 1\}$, *then* $\mathbb{F}_I := \mathbb{F}_1 \cup \mathbb{F}_2 \cup \{\infty\}$ *is a proper fundamental domain for* $SL_2(\mathbb{Z})$.

Proof. See [Ra 77], p. 51, Theorem 2.4.1. □

2.2 Stabilizers and fixed points

2.2.1 Definition. *Let* Γ *be a subgroup of* $SL_2(\mathbb{Z})$.

(a) Given a matrix $T \in \Gamma$, *a point* $z \in \overline{\mathbb{H}}$ *is called a* fixed point *of* T *iff.* $Tz = z$.

(b) $z \in \overline{\mathbb{H}}$ *is called a* fixed point *of* Γ *iff. there exists a* $T \neq \pm Id \in \Gamma$ *having* z *as a fixed point.*

(c) The stabilizer *of* $z \in \overline{\mathbb{H}}$, Γ_z *is defined to be the set of all such* $T \in \Gamma$ *having* z *as a fixed point. It is a subgroup of* Γ *(direct computation!). The homogeneous version will be referred to as* $\widehat{\Gamma}_z = \Phi(\Gamma_z) = \Gamma_z/\{(+-)Id\}$.

Before we start with the main part of this section, we define some special matrices in $\mathrm{SL}_2(\mathbb{Z})$ that we will need regularly and compute their fixed points.

2.2.2 Definition.

$$U := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad V := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$P := VU = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad P^2 = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$$

We have

$$\begin{aligned} Vz = z &\iff \frac{-1}{z} = z \\ &\iff -1 = z^2 \\ &\iff z = \pm i \end{aligned}$$

and analogously $Pz = z \iff z^2 + z + 1 = 0$. This equation has the solution $\varrho := e^{2\pi i/3}$ because

$$\begin{aligned}\varrho^2 + \varrho + 1 &= e^{\frac{2\pi i}{3}2} + e^{\frac{2\pi i}{3}1} + e^{\frac{2\pi i}{3}0} \\ &= \text{sum over the third roots of unity} \\ &= 0 \end{aligned} \tag{2.1}$$

by writing the sum as a finite geometric series. Since the coefficients are real, the complex conjugate $\overline{\varrho}$ is the second solution. As we will only be interested in fixed points in $\overline{\mathbb{H}}$, the solutions $-i, \overline{\varrho}$ will be unimportant to us.

The fixed points of P are therefore ϱ and $\overline{\varrho}$ and those of V are $\pm i$. Since $+P$ and $-P$, $+V$ and $-V$ respectively, to act the same on $\mathbb{H}$, we also found the fixed points of $-P$ and $-V$. The same calculation as above shows that $\pm P^2$ also has the fixed points ϱ and $\overline{\varrho}$.

We will now formulate our main goal in this section

2.2.3 Theorem. *The stabilizers of the points $\infty, i, \varrho \in \overline{\mathbb{H}}$ are given by*

$$SL_2(\mathbb{Z})_\infty = \{\pm U^k \mid k \in \mathbb{Z}\}$$

$$SL_2(\mathbb{Z})_i = \{\pm Id, \pm V\}$$

$$SL_2(\mathbb{Z})_\varrho = \{\pm Id, \pm P, \pm P^2\}$$

Summarized we have for arbitrary $z \in \overline{\mathbb{H}}$:

$$SL_2(\mathbb{Z})_z = \begin{cases} L\{\pm U^k \mid k \in \mathbb{Z}\}L^{-1} = \langle -Id, LUL^{-1}\rangle & \text{if } z = L.\infty \text{ for } L \in SL_2(\mathbb{Z}) \\ L\{\pm Id, \pm V\}L^{-1} = \langle -Id, LVL^{-1}\rangle, & \text{if } z = L.i \text{ for } L \in SL_2(\mathbb{Z}) \\ L\{\pm Id, \pm P, \pm P^2\}L^{-1} = \langle -Id, LPL^{-1}\rangle, & \text{if } z = L.\varrho \text{ for } L \in SL_2(\mathbb{Z}) \\ \{+Id, -Id\} = \langle +Id, -Id\rangle & \text{else} \end{cases} \tag{2.2}$$

in particular the stabilizers are never *cyclic. For the homogeneous versions we have*

$$\widehat{SL_2(\mathbb{Z})}_z = \begin{cases} \langle \Phi(LUL^{-1})\rangle & \text{if } z = L.\infty \text{ for some } L \in SL_2(\mathbb{Z}) \\ L\{\Phi(Id), \Phi(V)\}L^{-1}, & \text{if } z = L.i \text{ for some } L \in SL_2(\mathbb{Z}) \\ L\{\Phi(Id), \Phi(P), \Phi(P^2)\}L^{-1}, & \text{if } z = L.\varrho \text{ for some } L \in SL_2(\mathbb{Z}) \\ \{\Phi(Id)\} & \text{else} \end{cases} \tag{2.3}$$

in particular all homogeneous stabilizers are *cyclic.*

Proof. Step 1: Figuring out the potential fixed points. For doing this we need a preparatory lemma.

2.2.4 Lemma. *Given* $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *then either* $|\operatorname{Tr} T| > 2$ *– then* T *is of infinite order and its fixed points are in* $\mathbb{R} \setminus \mathbb{Q}$ *or* T *is conjugate to* $S = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ *where* S *is as shown in Table 1.*

Table 1: Possible conjugate matrices for T

$t = \alpha + \delta = a + d$	α	β	γ	δ	S
0	0	∓ 1	± 1	0	$\pm V$
1	0	∓ 1	± 1	1	$P, -V^{-1}P^2V$
	1			0	$-P^2, V^{-1}PV$
-1	0	∓ 1	± 1	-1	$-P, V^{-1}P^2V$
	-1			0	$P^2, -V^{-1}PV$
2	1	k	0	1	U^k
-2	-1	$-k$	0	-1	$-U^k$

Proof. See [Ra 77], Thm. 1.2.3, p. 10. □

Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}), T \neq Id$ be given and let $z \in \overline{\mathbb{H}}$ be a fixed point of T. We claim that

$$\text{fixed points of } \mathrm{SL}_2(\mathbb{Z}) \text{ in } \overline{\mathbb{H}} \subset [\infty] \cup [\varrho] \cup [i] \tag{2.4}$$

where $\varrho = e^{2\pi i/3}$.

First we want to eliminate the case where $c = 0$. If $c = 0$ then $a = d = \pm 1$ and $T = +U^k$ or $T = -U^k$ for some $k \in \mathbb{Z}$ where $k = \pm b$. In both cases, $Tz = z$ means $z \pm d = z$ which implies $z = \infty$. The case $d = 0$ cannot occur as $T \neq Id$. Consequently, (2.4) is correct for these fixed points. From now on we assume $c \neq 0$. We then rewrite the equation $Tz = z$ as

$$\begin{aligned} Tz = z \iff & \frac{az + b}{cz + d} = z \\ \iff & az + b = cz^2 + dz \\ \iff & cz^2 + (d - a)z - b = 0 \end{aligned}$$

Since $c \neq 0$, we find the solutions of this polynomial equation by the p/q-Formula to be:

$$z = \frac{(a - d) \pm \sqrt{\operatorname{Tr}(T)^2 - 4}}{2c}$$

The nature of the roots depends on the sign of the expression under the root.
1st case: $\mathrm{Tr}(T) = 0$

By Lemma 2.2.4, T a conjugate of $\pm V$ so there is some $L \in \mathrm{SL}_2(\mathbb{Z})$ with $T = \pm L^{-1}VL$. We obtain

$$\begin{aligned} Tz = z &\iff (\pm L^{-1}VL).z = z \\ &\iff \pm V.L.z = L.z \\ &\iff L.z \;\text{ is a fixed point of } \pm V \\ &\iff L.z = \pm i \\ &\iff z \in [i] \end{aligned}$$

$z = -i$ can never happen as $z \in \overline{\mathbb{H}}$ and so $Lz \in \overline{\mathbb{H}}$ but $-i \notin \overline{\mathbb{H}}$.
2nd case: $\mathrm{Tr}(T) = \pm 1$

Completely analogously, T is conjugate to $\pm P$ or $\pm P^2$ or $\pm V^{-1}PV$ ($\pm V^{-1}P^2V$ respectively) which is in turn conjugate to P (P^2 respectively) so $z \in [\varrho]$ in this case (as above, $Lz = \overline{\varrho}$ can never happen).
3rd case: $\mathrm{Tr}(T) = \pm 2$ Completely analogously, T is conjugate to $\pm U^k$ so $Lz = \infty$ by the $(c = 0)$–case or in other words: $z \in [\infty]$.
4th case: $|\,\mathrm{Tr}(T)| > 2$ We show that for $Tz = z$ we have $z \in \mathbb{R} \setminus \mathbb{Q}$ so $z \notin \overline{\mathbb{H}}$ (i.e. contradiction to the assumption $z \in \overline{\mathbb{H}}$). Note that

$$z \in \mathbb{R} \setminus \mathbb{Q} \iff \sqrt{(a+d)^2 - 4} \in \mathbb{R} \setminus \mathbb{Q} \iff (a+d)^2 - 4 \text{ is not a square}$$

the last " $\iff$ " is shown precisely as one shows that $\sqrt{2} \notin \mathbb{Q}$. For if k is a positive integer with prime factor decomposition $p_1^{e_1} \cdots p_r^{e_r}$ with, say e_1 odd, then

$$\begin{aligned} \sqrt{k} \in \mathbb{Q} \Rightarrow \exists \alpha, \beta \in \mathbb{Z} \;\; \sqrt{k} = \frac{\alpha}{\beta} \;\text{ with } \gcd(\alpha, \beta) = 1 & \\ \Rightarrow \beta^2 p_1^{e_1} \cdots p_r^{e_r} = \alpha^2 & \\ \Rightarrow \beta = p_1^x \cdots \;,\;\; \alpha = p_1^y \cdots \text{ and } 2x + e_1 = 2y & \\ \Rightarrow x \neq 0 \text{ as otherwise } 0 \equiv 2y = 2x + e_1 = e_1 \bmod 2 & \\ \text{(Contradiction as } e_1 \text{ is odd.)} & \\ \Rightarrow p_1 \text{ divides } \alpha \text{ and } \beta & \\ \Rightarrow \text{Contradiction to } \gcd(\alpha, \beta) = 1 & \end{aligned}$$

So all we have to show is that $(a+d)^2 - 4$ is not a square. Let $t := (a+d) = \mathrm{Tr}(T)$ and assume for a moment this was the case, then by assumption $t^2 - 4 > 0$ so there was an $x \in \mathbb{Z} \setminus \{0\}$ with $t^2 - 4 = x^2 \Rightarrow$

$(t-x)(t+x) = t^2 - x^2 = 4$. The only possible ways to factor the number 4 in $\mathbb{Z}$ are given by $((t-x),(t+x)) \in \{(\pm 1, \pm 4), (\pm 2, \pm 2), (\pm 4, \pm 1)\}$. By adding both equations we obtain in the first case that $2t \in \{-5, -3, 3, 5\}$, i.e. $t \in \{-5/2, -3/2, 3/2, 5/2\}$ which clearly contradicts the fact that $t \in \mathbb{Z}$. The third case is shown analogously and in the second case we obtain that $x = 0$, again by adding both equations. Thus, $(a+d)^2 - 4$ is not a square and therefor $z \in \mathbb{R} \setminus \mathbb{Q}$.

In particular we have shown $z \notin \mathbb{R} \setminus \mathbb{Q} \cup [\infty] \cup [\varrho] \cup [i] \Rightarrow \Gamma_z = \{+Id, -Id\}$, i.e. the fourth line of the claimed case-distinction is correct.
A direct computation shows that, since we have used " $\iff$ " only, we can do all calculations backwards and obtain moreover

$$\text{fixed points of } \mathrm{SL}_2(\mathbb{Z}) \text{ in } \overline{\mathbb{H}} = [\infty] \cup [\varrho] \cup [i]$$

Step 2: The fixed points in the orbits of i, ϱ and ∞. We analyze the nontrivial fixed points a little further.

2.2.5 Lemma. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ and x, y be two congruent points modulo Γ in $\overline{\mathbb{H}}$, such that there is an $L \in \Gamma$ with $Lx = y$, then*

$$\Gamma_y = L \Gamma_x L^{-1}$$

Consequently, for the homogeneous versions (as Φ is multiplicative):

$$\widehat{\Gamma}_y = \Phi(\Gamma_y) = \Phi(L\Gamma_x L^{-1}) = L\Phi(\Gamma_x)L^{-1} = L\widehat{\Gamma}_x L^{-1}$$

Proof.

$$\begin{aligned} T \in \Gamma_y &\iff Ty = y \\ &\iff TLx = Lx && (y = Lx) \\ &\iff L^{-1}TLx = x \\ &\iff L^{-1}TL \in \Gamma_x \\ &\iff T \in L\Gamma_x L^{-1} \end{aligned}$$

□

3rd Step: Putting it all together. Step 1 tells us that we only have to analyze the stabilizers of points $x \in [\infty] \cup [\varrho] \cup [i]$. Step 2 tells us that it suffices to compute the stabilizers of the points i, ϱ and ∞ itself and thus,

this is what we will do now: The "$\supset$"-direction has been done right after definition 2.2.2 so it remains to show:
$\underline{\mathrm{SL}_2(\mathbb{Z})_\infty \subset \{\pm U^k \mid k \in \mathbb{Z}\}}$: Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ so that $a/c = T.\infty = \infty$. The only possibility is $c = 0$ and hence $a = d = \pm 1$ and thus $T = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) = U^c$.
$\underline{\mathrm{SL}_2(\mathbb{Z})_i \subset \{\pm Id, \pm V\}}$: Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ so that $(ai+b)/(ci+d) = T.i = i$. Then $ai + b = i(ci + d) = -c + di$. Comparing real- and imaginary part yields $b = -c, a = d$ so $T = \left(\begin{smallmatrix} a & b \\ -b & a \end{smallmatrix}\right)$. Since $\pm 1 = det(T) = a^2 + b^2$ the only possible solutions for a and b are $a = 0, b = \pm 1$ or vice versa, i.e. $T = \pm V$ or $T = \pm Id$.
$\underline{\mathrm{SL}_2(\mathbb{Z})_\varrho \subset \{\pm Id, \pm P, \pm P^2\}}$: Let $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ so that $(a\varrho+b)/(c\varrho+d) = T.\varrho = \varrho$. Then $a\varrho + b = \varrho(c\varrho + d)$ and since $\varrho^2 = -(\varrho + 1)$, cf (2.1), $a\varrho+b = \varrho(-c+d)-c$. Writing $\varrho = \alpha + i\beta$ and comparing real- and imaginary part yields

$$a\alpha + b = (-c + d)\alpha - c \text{ and } a\beta = (-c + d)\beta \tag{2.5}$$

hence, as $\beta \neq 0$, $a = (-c + d)$. Substituting this into the first equation in (2.5), we obtain $b = -c$ and thus $T = \left(\begin{smallmatrix} (b+d) & b \\ -b & d \end{smallmatrix}\right)$. The equation on the determinant then reads $1 = det(T) = bd + b^2 + d^2$. In the case where $bd \geq 0$ we have $1 = bd + b^2 + d^2 \geq b^2 + d^2$ so $b = 0, d = \pm 1$ (i.e. $T = \pm Id$) or $b = \pm 1, d = 0$ (i.e. $T = \pm P^2$) are the only solutions. In the case where $bd < 0$ we have $1 = bd + b^2 + d^2 > bd + bd + b^2 + d^2 = (b + d)^2$ thus $|b + d| < 1$ and consequently $b + d = 0$. Substituting this into the determinant equation yields $d = \pm 1$ (and therefor $b = \mp 1$ and thus $T = \pm P$).
Now we apply Lemma 2.2.5 to $x = i, \varrho, \infty$ and $y = L.x$ in the orbit of x. According to what we have just computed, this yields

$$\Gamma_y = \Gamma_{L.z} = L\Gamma_x L^{-1}$$

Since Φ is multiplicative, the assertion concerning the homogeneous stabilizers $\widehat{\Gamma}_z$ follows by

$$\Phi(\langle -Id, X\rangle) = \langle \Phi(Id), \Phi(X)\rangle = \langle \widehat{X}\rangle$$

substituted for $X = V, P$ or U. $\square$

Now we want to characterize the homogeneous stabilizers of a subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ with a finite index in the same way. We remark that by definition, $\Gamma_z = \mathrm{SL}_2(\mathbb{Z})_z \cap \Gamma$.

We first note that the stabilizers Γ_z ($\widehat{\Gamma}_z$ respectively) are subgroups of $\mathrm{SL}_2(\mathbb{Z})_z$ ($\widehat{\mathrm{SL}_2(\mathbb{Z})}_z$ respectively) by construction. For a finer analysis of the stabilizer of the points $L.\infty$ we need some preparation:

2.2.6 Lemma. *Let H be a subgroup of a group G having a finite index, then for each element $S \in G$, there is a finite natural number n such that $S^n \in H$ and a unique smallest natural number n_S with $S^{n_S} \in H$.*

Proof. Consider the right cosets $[Id], [S], [S^2], [S^3], ...$ in G/H. Since the index is finite, G/H is finite so at some point, two of the cosets must be the same, say $[S^i] = [S^j]$ for some $i, j \in \mathbb{N}$ and after switching i, j we may assume $i < j$. $[S^i] = [S^j]$ implies that there is a $T \in H$ such that $S^i = TS^j$. Multiplying this equation with S^{-i} yields $Id = TS^{j-i}$ so $S^{j-i} = T^{-1} \in H$ means that $0 \leq j - i$ is one possible n. Now $n_S := \min\{n \in \mathbb{N} \mid S^n \in H\}$ exists as the set is not empty. □

2.2.7 Lemma. *Let G be a cyclic group, say $G = \langle S \rangle$ and let H be a subgroup of G having a finite index, then H is also cyclic and*

$$H = \langle S^{n_S} \rangle, \quad G/H = \{[Id], [S], ..., [S^{n_s} - 1]\}$$

and in particular $[G : H] = n_S$ where n_S is as in the lemma above

Proof. Analogously to the proof that subgroups of $\mathbb{Z}$ are of the structure $N\mathbb{Z}$ for some $N \in \mathbb{N}$. □

2.2.8 Corollary. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having a finite index, then for each element $S \in \widehat{SL_2(\mathbb{Z})}$ there is a unique finite smallest natural numbers $\widehat{n}$ such that $\widehat{S}^{\widehat{n}} \in \widehat{\Gamma}$.*

2.2.9 Corollary. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having a finite index, then for each element $S \in SL_2(\mathbb{Z})$ there is a unique finite smallest natural numbers n such that $S^n \in \widehat{\Gamma}$.*

Proof. Substitute $H = \Gamma$ and $G = \mathrm{SL}_2(\mathbb{Z})$ ($H = \widehat{\Gamma}, G = \widehat{\mathrm{SL}_2(\mathbb{Z})}$ respectively) and apply the above lemma. □

2.2.10 Theorem. *Let Γ be a subgroup having finite index in $SL_2(\mathbb{Z})$, then the* homogeneous *stabilizers of all $z \in \overline{\mathbb{H}}$ are cyclic. More precisely:*

$$\widehat{\Gamma}_z = \begin{cases} \langle \Phi(LU^{n_L}L^{-1}) \rangle & \text{if } z = L.\infty \\ \{\Phi(Id), \Phi(LVL^{-1})\}, & \text{if } z = L.i \text{ and } \Phi(LVL^{-1}) \in \widehat{\Gamma} \\ \{\Phi(Id), \Phi(LPL^{-1}), \Phi(LP^2L^{-1})\}, & \text{if } z = L.\varrho \text{ and } \Phi(LPL^{-1}) \in \widehat{\Gamma} \\ \{\Phi(Id)\} & \text{otherwise} \end{cases}$$

Proof. If $z \notin [\infty] \cup [i] \cup [\varrho]$ but $z \in \overline{\mathbb{H}}$, then by equation (2.4), $\mathrm{SL}_2(\mathbb{Z})_z = \{\pm Id\}$ and therefore $\widehat{\Gamma}_z \subseteq \widehat{\mathrm{SL}_2(\mathbb{Z})}_z = \{\Phi(\pm Id)\}$ hence $\widehat{\Gamma}_z = \{\widehat{Id}\}$.

The case $z = L.\infty$: Given $z = L.\infty$, by Lemma 2.3, the stabilizer of $\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L.\infty}$ is given by $\langle \Phi(LUL^{-1}) \rangle$ with generator $S := S_L := \Phi(LUL^{-1}) = \widehat{L}\widehat{U}\widehat{L}^{-1}$. For readability we identify L, U, Id with $\Phi(L), \Phi(U), \Phi(Id)$ from now on for the rest of this case, i.e. whenever we write L, U, Id we actually mean $\Phi(L), \Phi(U), \Phi(Id)$. If we apply Corollary 2.2.8 to the generator S of the stabilizer we obtain unique smallest $\widehat{n_L}$ such that $S^{\widehat{n_L}} = LU^{\widehat{n_L}}L^{-1} \in \widehat{\Gamma}$ and

$$\widehat{\mathrm{SL}_2(\mathbb{Z})}_z / \widehat{\Gamma}_z = \{[Id], [LUL^{-1}], ..., [LU^{\widehat{n_L}}L^{-1}]\}$$

(where $[\cdot]$ denotes the class of "$\cdot$" in the quotient $\widehat{\mathrm{SL}_2(\mathbb{Z})}_z / \widehat{\Gamma}_z$). Further, by Lemma 2.2.7, we have

$$[\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L.\infty} : \widehat{\Gamma}_{L.\infty}] = \widehat{n_L}$$

and

$$\widehat{\Gamma}_{L.\infty} = \{Id, LU^{\widehat{n_L}}L^{-1}, LU^{2\widehat{n_L}}L^{-1}, ...\} = \langle LU^{\widehat{n_L}}L^{-1} \rangle$$

The cases $z = L.i, z = L.\varrho$: Here we see that an 'all-or-nothing-principle' applies, meaning that either the stabilizer $\widehat{\Gamma}_z$ stays at its full size or collapses to $\{\Phi(Id)\}$. In the case $z = L.i$ either $\Phi(LVL^{-1})$ in $\widehat{\Gamma}$ (then the stabilizer has size 2 hence is the same as the stabilizer in the big group $\widehat{\mathrm{SL}_2(\mathbb{Z})}$) or $\Phi(LVL^{-1}) \notin \widehat{\Gamma}$, then the stabilizer must be a subset of $\widehat{\mathrm{SL}_2(\mathbb{Z})}_z \setminus \{\Phi(LVL^{-1})\} = \{\Phi(Id)\}$ hence it must actually be $\{\Phi(Id)\}$. Analogous arguments apply to $z \in [\varrho]$ because the difference between P and P^2 can be omitted as $\Phi(LPL^{-1}) \in \widehat{\Gamma} \iff \Phi(LP^2L^{-1}) \in \widehat{\Gamma}$ because in this case, squaring leads from one to the other (and back!). □

The 'all'-points in $\mathbb{H}$ in the 'all-or-nothing-principle' of the homogeneous stabilizers (or more precisely: their orbits) will be of interest later on so we define:

2.2.11 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having finite index, then we define the sets*

$$\mathbb{E}_2(\Gamma) := \{[\![z]\!] \subset \mathbb{H} \mid z = L.i \text{ for some } L \in SL_2(\mathbb{Z}) \text{ and } \Phi(LVL^{-1}) \in \widehat{\Gamma}\}$$

$$\mathbb{E}_3(\Gamma) := \{[\![z]\!] \subset \mathbb{H} \mid z = L.\varrho \text{ for some } L \in SL_2(\mathbb{Z}) \text{ and } \Phi(LPL^{-1}) \in \widehat{\Gamma}\}$$

$$\mathbb{E}(\Gamma) := \mathbb{E}_2(\Gamma) \cup \mathbb{E}_3(\Gamma), \quad \mathbb{E}_2 = \mathbb{E}_2(SL_2(\mathbb{Z})), \ \mathbb{E}_3 = \mathbb{E}_3(SL_2(\mathbb{Z})), \ \mathbb{E} = \mathbb{E}_2 \cup \mathbb{E}_3$$

Note that these sets do not contain points as it but rather their orbits modulo the smaller group. The reason for this is that the the describing quantities of the "behavior" of f do not depend on the concrete point but rather on the orbit. Also not that proposition 2.3.5 will tell us later that for different representatives z_1, z_2 of the same fixed-point orbit, the generators of the stabilizers do only vary by multiplication with elements in $\widehat{\Gamma}$ so whether or whether not $\Phi(LVL^{-1}) \in \widehat{\Gamma}$ ($\Phi(LPL^{-1}) \in \widehat{\Gamma}$ respectively) does not depend on the concrete representative, i.e. $\mathbb{E}_2(\Gamma), \mathbb{E}_3(\Gamma)$ are well definitions. Described in a visual way, let $\widehat{\mathcal{R}} = \{\widehat{R_1}, ..., \widehat{R_n}\}$ be an RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$, then the orbit $[i]$ decomposes into a disjoint union of sevaral 'smaller' orbits

$$[i] = \mathrm{SL}_2(\mathbb{Z}).i = \widehat{\Gamma}.\widehat{\mathcal{R}}.i = \widehat{\Gamma}.\{\widehat{R_1}, ..., \widehat{R_n}\}.i = [\![R_1.i]\!] \,\dot{\cup}\, ... \,\dot{\cup}\, [\![R_n.i]\!]$$

and $\mathbb{E}_2(\Gamma)$ collects all those 'smaller' orbits for which the stabilizer does not collapse.

2.2.12 Remark. *Note that in this case we used the homogeneous versions of the groups all the time. This is very important in this case: by the usage of the term "fixed point" we mean that there is an action in the set of actions induced by the subgroup that leaves the point fixed. We do not want the matrix in Γ to have a certain sign. For example: we will show in Theorem 4.3.1 that the subgroup $\Gamma_1(3)$ possesses only one fixed point (orbit), namely the orbit of $Q = 1/2 + i\sqrt{3}$. The matrix that leaves Q fixed is $T = \left(\begin{smallmatrix} -2 & 1 \\ -3 & 1 \end{smallmatrix}\right)$. T is conjugate to $-P$ but we will also show in 4.3.1 that there is* no *matrix $T' \in \Gamma_1(3)$ conjugate to $+P$. As in the case for $SL_2(\mathbb{Z})$, the inhomogeneous stabilizers are not cyclic but as they are uninteresting to us we will skip the exact analysis of them.*

2.3 Cusps and their width

2.3.1 Definition. *(a) For Γ a subgroup of $SL_2(\mathbb{Z})$, $z \in \overline{\mathbb{H}}$, the set* $\mathrm{tr}_\Gamma(z) := \{[\![w]\!] \mid w \in [z]\}$ *is called the* trace of z *(modulo Γ). Again, we will suppress Γ and will just write* $\mathrm{tr}(z)$ *in place of* $\mathrm{tr}_\Gamma(z)$.

(b) Representatives of the equivalence classes in $\mathrm{tr}(\infty)$ *are called* cusps *and their class in* $\mathrm{tr}(\infty)$ *is called* cusp orbit. *We will also denote* $\mathrm{cu}(\Gamma) := \mathrm{tr}(\infty)$.

It is easy to see that whether or whether not $w \in [z]$ does not depend on the concrete representative of $[\![w]\!]$ so $\mathrm{tr}(z)$ is well-defined. Also note that a direct computation shows that after choosing an RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$, $\widehat{\mathcal{R}} = \{\widehat{R_1}, ..., \widehat{R_n}\}$, we have

$$\{[\![w]\!] \mid w \in [z]\} = \mathrm{SL}_2(\mathbb{Z}).z/\!\approx\; = \widehat{\Gamma}.\widehat{\mathcal{R}}/\!\approx\; = \widehat{\Gamma}.R_1.z \,\dot\cup\, ... \,\dot\cup\, \widehat{\Gamma}.R_n.z,$$

i.e. for knowing the trace of z it suffices to translate z with an RRS and then consider the orbits of all these finitely many RRS-translations of z.

2.3.2 Example. *In $SL_2(\mathbb{Z})$, the cusps are given by $\mathbb{P}$ (recall that $\mathbb{P} = \mathbb{Q} \cup \{\infty\}$). This can be shown as follows: Let any a'/c' be in $\mathbb{Q}$. By canceling out every possible divisor we achieve $a'/c' = a/c$ and $\gcd(a,c) = 1$. From general algebra we know that then, there are integers b, d such that $ad - bc = \gcd(a,c) = 1$, i.e. the matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ is in $SL_2(\mathbb{Z})$. Now $a'/c' = a/c = T.\infty \in SL_2(\mathbb{Z}).\infty = [\infty] \subseteq \mathbb{Q} \cup \{\infty\} = \mathbb{P}$.*

2.3.3 Example. *The subgroup $\Gamma_0(2)$ possesses two cusps and the trace of ∞ is $\{[\![0]\!], [\![\infty]\!]\}$ and $0, \infty$ are incongruent modulo $\Gamma_0(2)$. Justification: to show: $\{[\![w]\!] \mid w \in [\infty]\} = \{[\![0]\!], [\![\infty]\!]\}$. "$\supseteq$": $0, \infty \in [\infty]$ as $Id.\infty = \infty$ and $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right).\infty = 0$. "$\subseteq$": let $w \in [\infty] = \mathbb{P}$ s.t. $w \neq \infty$, then $w = x/y$ for some $x, y \in \mathbb{Z}$ and by canceling out all common divisors we may assume $\gcd(x,y) = 1$ so there are integers u, v such that $ux + vy = \gcd(x,y) = 1$. 1st case: y is even, then $T = \left(\begin{smallmatrix} x & -v \\ y & u \end{smallmatrix}\right) \in \Gamma_0(2)$ as y is even and $\det(T) = xu - y(-v) = xu + yv = 1$ and furthermore, $T.\infty = x/y = w$ so $w \in [\![\infty]\!]$. 2nd case: y is odd. We may assume that u is even as otherwise we can also set $u' = u - y, v' = v + x$ and still solve $u'x + v'y = 1$ because $u'x + v'y = (u-y)x + (v+x)y = ux + vy + xy - xy = ux + vy = 1$ but this time, $u' \equiv u - y \equiv 1 - 1 \equiv 0 \mod 2$. Then, for $T = \left(\begin{smallmatrix} v & x \\ -u & y \end{smallmatrix}\right)$ we have $T.0 = \frac{v0+x}{(-u)0+y} = \frac{x}{y} = w$ and $T \in \Gamma_0(2)$ as $\det(T) = vy - (-u)x = ux + vy = 1$ and u is even. Consequently, $w = T.0 \in [\![0]\!]$. Moreover, 0 and ∞ are incongruent modulo $\Gamma_0(2)$. Indeed, given a matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$ so that c is even, then $T.0 = \frac{b}{d} = \infty$ implies $d = 0$ so in particular c and d are even so that $2 \mid c, 2 \mid d \Rightarrow 2 \mid ad - bc = \det(T) = 1$ so $2 \mid 1 \Rightarrow$ contradiction.*

Since $\mathbb{P} = [\infty]$, in $\mathrm{SL}_2(\mathbb{Z})$, every cusp has the form $L.\infty$ for some $L \in \mathrm{SL}_2(\mathbb{Z})$. We now want to define a quantity called 'width' of the cusp $L.\infty$. Fix some subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$ having a finite index. Recall from section 2.2, in particular Theorem 2.2.10, that the stabilizer $\widehat{\Gamma}_z$ is cyclic for all $z \in \overline{\mathbb{H}}$ and if $\widehat{\Gamma}_z = \langle \pm S \rangle$ then some power of S must lie in $\widehat{\Gamma}$ (see Theorem 2.2.8) or reformulated as matrices: there is an $n_S \in \mathbb{N}$ with either $+S^{n_S} \in \Gamma$ or $-S^{n_S} \in \Gamma$.

2.3.4 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having a finite index and $L.\infty$ a cusp. Let $S := LUL^{-1}$, then $\widehat{S} = \Phi(S)$ is a generator of the cyclic homogeneous stabilizer according to Theorem 2.2.10. By Corollaries 2.2.8, 2.2.9, there are unique minimal naturals $\widehat{n_L}, n_L$ with $\widehat{S}^{\widehat{n_L}} \in \widehat{\Gamma}$ and $S^{n_L} \in \Gamma$. We define*

1. $\widehat{n_L} = \min\{n \in \mathbb{N} \mid \Phi(LU^nL^{-1}) = \widehat{S}^n \in \widehat{\Gamma}\}$ *to be the* homogeneous width of the cusp $L.\infty$.
2. $n_L = \min\{n \in \mathbb{N} \mid LU^nL^{-1} \in \Gamma\}$ *to be the* inhomogeneous width of the cusp $L.\infty$.

There is a detail hidden in this definition. For a given cusp $c = L.\infty$, we want to write $\widehat{n_c}, n_c$ and not $\widehat{n_L}, n_L$. Until now, the width has the possibility to "look how ∞ got there", i.e. we have only defined the width for given cusp *and* transformation matrix that brings ∞ to the cusp. In $\Gamma_1(3)$ for example, there are many different ways how to get from ∞ to 0, i.e. $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).\infty = 0$ but also $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 17 \end{smallmatrix}\right).\infty = 0$. The arising question is: are $\widehat{n_0}, n_0$ as defined with both matrices the same? The following will answer this positively:

2.3.5 Proposition. *Let $L, M \in SL_2(\mathbb{Z})$ both mapping infinity to the cusp $c = L.\infty = M.\infty$, then both – homogeneous and inhomogeneous width – do not depend on L or M but rather only on the cusp c, i.e. $\widehat{n_L} = \widehat{n_M}$ and $n_L = n_M$, so "$n_c := n_L$ for any $L \in SL_2(\mathbb{Z})$ with $c = L.\infty$" is well-defined.*

Proof. Since $L.\infty = M.\infty$, we have $L^{-1}M.\infty = \infty$ so $L^{-1}M \in \mathrm{SL}_2(\mathbb{Z})_\infty$ and thus there is a $k \in \mathbb{Z}$ such that $L^{-1}M = \pm U^k$. Then the matrix S mentioned in the definition of n_L and n_M is actually the same. In the "+" case we have

$$S_M = MUM^{-1} = LU^kUU^{-k}L^{-1} = LUL^{-1} = S_L$$

so in particular for any $n \in \mathbb{N}$, $S_M^n \in \Gamma \iff S_L^n \in \Gamma$ so $n_L = n_M$ in this case. In the "$-$" case we have

$$S_M = MUM^{-1} = (-L)U^kUU^{-k}(-L^{-1}) = \cancel{(-1)^2}LUL^{-1} = S_L$$

so in particular for any $n \in \mathbb{N}$, $S_M^n \in \Gamma \iff S_L^n \in \Gamma$ so $n_L = n_M$ is also valid here. In the homogeneous situation the different sign does not matter at all as

$$L^{-1}M = \pm U^k \Rightarrow \Phi(M) = \Phi(L)\Phi(U)^k$$

anyway. ∎

Although this may look a little complicated, it eases up the main proofs afterwards because for a cusp $c \in \mathbb{P}$, we directly want to consider an arbitrary (but fixed) matrix $L \in \mathrm{SL}_2(\mathbb{Z})$ that brings ∞ to c and the notation n_L already includes this. Remark that we follow the notation of [Ra 77] here.

The distinction between $\widehat{n_L}$ and n_L is necessary as in some situations they are unequal. As this case will have an impact on the calculations afterwards we will separate these cases more clearly:

2.3.6 Definition. *Let $c = L.\infty$ be a cusp, then c is called a* regular cusp *if $n_c = \widehat{n_c}$, otherwise it is called an* irregular cusp.

2.3.7 Example. *Irregular cusps exist: Take $\Gamma = \Gamma_1(4)$ then for the cusp $1/2 = L.\infty = \left(\begin{smallmatrix} 1 & 1 \\ 2 & 3 \end{smallmatrix}\right).\infty$, we have $S := LUL^{-1} = \left(\begin{smallmatrix} -1 & 1 \\ -4 & 3 \end{smallmatrix}\right)$ so $S^1 \notin \Gamma$ but $-S \in \Gamma$ so $\widehat{n_L} = 1$ but $n_L > 1$.*

2.3.8 Remark. *If $n_L \neq \widehat{n_L}$, then $n_L = 2\widehat{n_L}$. Set $S = LUL^{-1}$, then by definition of $\widehat{n_L}, \pm \widehat{S^{\widehat{n_L}}} \in \widehat{\Gamma}$ but $+S^{\widehat{n_L}} \in \Gamma$ is impossible as then $n_L \leq \widehat{n_L}$ (and $\widehat{n_L} \leq n_L$ is clear anyway) but $n_L \neq \widehat{n_L}$ was assumed, so $-S^{\widehat{n_L}} \in \Gamma$. Consequently, $S^{2\widehat{n_L}} = (-S^{\widehat{n_L}})^2 \in \Gamma \cdot \Gamma = \Gamma$, therefore $n_L \leq 2\widehat{n_L}$. Assume $n_L < 2\widehat{n_L}$ i.e. $n_L = \widehat{n_L} + r$ for $0 < r < \widehat{n_L}$, then $S^{n_L} \in \Gamma, -S^{\widehat{n_L}} \in \Gamma \Rightarrow (-S^{\widehat{n_L}})^{-1} \in \Gamma$ so $-S^r = S^{\widehat{n_L}+r} \cdot (-1) \cdot S^{-\widehat{n_L}} = S^{n_L} \cdot (-S^{\widehat{n_L}})^{-1} \in \Gamma$ and thus $\widehat{n_L} \leq r < \widehat{n_L}$. Contradiction.*

2.4 Modular forms

From now on we will assume that $k \in \mathbb{Z}$ is a fixed integer. We assume this because in general (for $k \in \mathbb{R}$), $z, w \in \mathbb{C}, (zw)^k \neq z^k w^k$ but of course, for integers this works and this will ease up some of the computations below.

2.4.1 Definition. *For $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$ and $z \in \overline{\mathbb{H}}$ we define $(T : z)$ to be $cz + d$ and $\mu(T, z) := (T : z)^k$.*

2.4.2 Proposition. *[ST-identity] For any $S, T \in SL_2(\mathbb{Z})$, $z \in \mathbb{C}$ we have $(ST : z) = (S : Tz)(T : z)$ and in particular for $S = -Id$ we have $(-T : z) = -(T : z)$.*

Proof. Direct computation. □

2.4.3 Definition (Modular form)**.** *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having a finite index. We define for any $L \in SL_2(\mathbb{Z})$ (not necessarily $L \in \Gamma$!) the L-transform of f to be $f_L : \mathbb{H} \to \mathbb{C}$, $f_L(z) = \frac{1}{\mu(L,z)} f(L.z)$. Each function $f : \mathbb{H} \to \mathbb{C}$ that possesses the properties*

(I) f is holomorphic on $\mathbb{H}$.

(II) f transforms under Γ: for all $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$, $f(T.z) = (cz+d)^k f(z)$.

(III) For every $L \in SL_2(\mathbb{Z})$, f_L has a Fourier expansion of the form

$$f_L(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z / n_L}$$

for suitable $(a_n)_{n \in \mathbb{N}} \subset \mathbb{C}$.

is called a modular form *for Γ of weight k.*

For a subgroup Γ of $SL_2(\mathbb{Z})$, the set of all such f of weight k will be referred to as $M(\Gamma, k)$ and the set of all modular forms in general (of arbitrary weight) will be denoted $M(\Gamma)$.

Throughout this document (if not explicitly defined otherwise), f will denote a modular form for a subgroup Γ.

We remark that if f is a modular form and $G \in \Gamma$, then $f_G \equiv f$. We now want to get a better understanding of condition (III). The following theorem states that (III) consists of the requirement that the coefficients start with a positive index (i.e. that $a_n = 0$ for $n < 0$) and not in the condition that a Fourier expansion exists, because this is an implication from conditions (I) and (II):

2.4.4 Theorem. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ of finite index and f be a function that satisfies condition (I) and (II), then*

(a) f_L is n_L-periodic, i.e. $f_L(z + n_L) = f_L(z) \quad \forall z \in \overline{\mathbb{H}}$

(b) There always exists a Fourier expansion of f_L of the form

$$f_L(z) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n z / n_L}$$

possibly containing an infinite number of terms carrying a negative index.

Proof. (a): Let $L = \left(\begin{smallmatrix} l_1 & l_2 \\ l_3 & l_4 \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and define $S = LU^{n_L}L^{-1}$. By definition of n_L, $S \in \Gamma$. Also note that $LU^{n_L} = SL$. With this we can write

$$
\begin{aligned}
f_L(z+n_L) &= f_L(U^{n_L}z) \\
&= \frac{1}{(L:U^{n_L}z)^k} f(LU^{n_L}z) \\
&= \frac{1}{(L:U^{n_L}z)^k} f(SLz) \\
&= \frac{1}{(L:U^{n_L}z)^k} (S:Lz)^k f(Lz) \\
&\qquad (f \text{ satisfies (II)}, \ S \in \Gamma) \\
&= \underbrace{\frac{1}{(L:U^{n_L}z)^k} (S:Lz)^k (L:z)^k}_{:=g(z)} \underbrace{\frac{1}{(L:z)^k} f(Lz)}_{=f_L(z)} \\
&= g(z) f_L(z)
\end{aligned}
$$

It remains to show that the function

$$
g : z \mapsto \frac{(S:Lz)^k (L:z)^k}{(L:U^{n_L}z)^k}
$$

is constantly one:

$$
\begin{aligned}
g(z) &= \frac{(S:Lz)^k (L:z)^k}{(L:U^{n_L}z)^k} \\
&= \left(\frac{(L(U^{n_L}L^{-1}):Lz)(L:z)}{(L:U^{n_L}z)}\right)^k && (S = LU^{n_L}L^{-1}) \\
&= \left(\frac{(L:U^{n_L}\cancel{(L^{-1}L)}z)(U^{n_L}L^{-1}:Lz)(L:z)}{(L:U^{n_L}z)}\right)^k && \text{by (2.4.2)} \\
&= \left(\frac{\cancel{(L:U^{n_L}z)}\overbrace{(U^{n_L}:L^{-1}Lz)}^{=0z+1=1}(L^{-1}:Lz)(L:z)}{\cancel{(L:U^{n_L}z)}}\right)^k && \text{by (2.4.2)} \\
&= [(L^{-1}:Lz)(L:z)]^k \\
&= [[(-l_3)Lz + l_1](L:z)]^k \\
&= \left(\left[(-l_3)\frac{l_1 z + l_2}{l_3 z + l_4} + l_1\right](L:z)\right)^k \\
&= \left(\left[\frac{\cancel{-l_3 l_1 z} - l_3 l_2 + \cancel{l_1 l_3 z} + l_1 l_4}{l_3 z + l_4}\right](L:z)\right)^k \\
&= \left(\frac{\det(L)}{L:z}(L:z)\right)^k \\
&= 1^k = 1
\end{aligned}
$$

(b): Consider the mapping $\alpha : z \mapsto e^{\frac{2\pi i z}{n_L}}$ then α maps $\mathbb{H}$ surjectively to $B_1(0) = \{t \in \mathbb{C} \mid |t| < 1\}$ (it 'increases' the density by 'pushing' $z \in \mathbb{H}$ to $t \in B_1(0)$) and we can make α continuous in the extended topology by setting $\alpha(\infty) = 0$ i.e. α "switches" the roles of 0 and ∞. One of the many "inverse" mappings is given by $\beta : t \mapsto n_L \operatorname{Log}(t)/2\pi i$ in the sense that $\alpha(\beta(t)) = t$. This mapping is neither holomorphic nor continuous because Log is discontinuous in all $z \in \mathbb{R}^-$. However: the map $G : B_1(0) \setminus \{0\} \mapsto \mathbb{H}, G(t) = f_L(\beta(t))$ is actually holomorphic: For $t_0 \in B_1(0)$ we first select a branch of the complex logarithm that exists in an open neighborhood U around t_0, Log_U. Then, the function $\beta_U : t \mapsto n_L \operatorname{Log}_U(t)/2\pi i$ and hence $G_U : t \mapsto f_L(\beta_U(t))$ is holomorphic on U as a composition of such. Now we use (a) to show that $G_U(t) = G(t)$ for all $t \in U$. For all $t \in U$ we know from complex analysis that $\operatorname{Log}_U(t) = \log_{\mathbb{R}} |t| + i \operatorname{ph}(t) + 2\pi i d = \operatorname{Log}(t) + 2\pi i d$

where $d = d_U \in \mathbb{Z}$ only depends on U. Hence

$$\begin{aligned} G_U(t) &= f_L(\beta_U(t)) \\ &= f_L\left(\frac{n_L \operatorname{Log}_U(t)}{2\pi i}\right) \\ &= f_L\left(\frac{n_L \operatorname{Log}(t)}{2\pi i} + \frac{\cancel{2\pi i} d n_L}{\cancel{2\pi i}}\right) \\ &= f_L\left(\frac{n_L \operatorname{Log}(t)}{2\pi i}\right) \qquad \text{(as } f_L \text{ is } n_L\text{-periodic and } d \in \mathbb{Z}) \\ &= G(t) \end{aligned}$$

So $G = G_U$ is locally holomorphic on U around t_0. The same arguments apply to arbitrary $t_0 \in B_1(0) \setminus \{0\}$ (using a localized branch Log_{t_0} of the complex logarithm for every t_0). From complex analysis, we now know that for the function G, a Laurent series exists around 0:

$$G(t) = \sum_{n=-\infty}^{\infty} a_n t^n$$

Note that we have by construction $G(t) = f_L(n_L \operatorname{Log}(t)/2\pi i)$ so if any $z \in \mathbb{H}$ is given, then

$$G(e^{2\pi i z/n_L}) = f_L(n_L \operatorname{Log}(e^{2\pi i z/n_L})/2\pi i) = f_L(n_L 2\pi i z/2\pi i n_L) = f_L(z)$$

i.e. in short $f_L(z) = G(\alpha(z))$. Hence:

$$f_L(z) = G(\alpha(z)) = \sum_{n=-\infty}^{\infty} a_n e^{\frac{2\pi i n z}{n_L}}$$

. □

2.4.5 Remark. *Note that in the computation of 2.4.4(a) we explicitly used the difference between n_L and $\widehat{n_L}$, i.e. in general, the function f_L is* **not** *$\widehat{n_L}$ periodic for if k is odd and $+S \notin \Gamma$ but $-S = -LU^{\widehat{n_L}}L^{-1} \in \Gamma$ (i.e. in the case where $n_L \neq \widehat{n_L}$), the step*

$$f(SLz) = (S : Lz)^k f_L(Lz)$$

is invalid as $+S$ is not in the subgroup anymore. Still, $-S \in \Gamma$ and $S, -S$ induce the same action so one could write

$$f(SLz) = f(-SLz) = (-1)^k (S : Lz)^k f_L(Lz)$$

and all in all we therefore obtain

$$f_L(z+\widehat{n_L}) = (-1)g(z)f_L(z) = -f_L(z)$$

so in particular, if k is odd, the only form being $\widehat{n_L}$ periodic is the zero form ($f_L(z) = f_L(z+\widehat{n_L}) = -f_L(z)$ implies $f_L(z) = 0$ for all $z \in \mathbb{H}$ and hence $f(z) = 0$ for all z as $z \mapsto Lz$ is a bijection).

2.4.6 Definition. *The functions G and β that have been used in the proof above for the cusp $L.\infty$ will be called G_L and β_L from now on, i.e.*

$$G_L : B_1(0) \setminus \{0\} \mapsto \mathbb{H}, \quad G_L(t) = f_L(h(t))$$

$$\beta_L : B_1(0) \setminus \{0\} \mapsto \mathbb{H}, \quad \beta_L(t) = \frac{n_L \operatorname{Log}(t)}{2\pi i}$$

Moreover we will write $q_L = e^{2\pi i z/n_L}$ in place of $\alpha_L(z)$ (in order to emphasis the role of $t = e^{2\pi i z/n_L}$ as a new variable) and $q = e^{2\pi i z}$.

The task of $\alpha_L(z)$ was to switch the positions of 0 and ∞. This can be made more precise:

2.4.7 Theorem. *Let f be a function satisfying conditions (I) and (II), then*

$$\begin{aligned}
& f \text{ satisfies condition (III)} \\
\iff & \text{for all cusps } L.\infty,\ G_L \text{ is holomorphic at } 0 \\
:\iff & f \text{ is holomorphic at } L.\infty \\
:\iff & f_L \text{ is holomorphic at } \infty \\
:\iff & f_L \text{ is continuous at } \infty \\
:\iff & \text{There is a } c \in \mathbb{C} \text{ s.t. for all sequences } (z_n)_{n\in N} \text{ with} \\
& \operatorname{Im}(z_n) \to \infty,\ \lim_{n\to\infty} f_L(z_n) = c
\end{aligned}$$

If these conditions are met, then a_0 of the Laurent series of G_L is actually c. For such sequences, we will write shorter $z_n \to i\infty$ and $f_L(i\infty) = c$.

Proof. "$\Rightarrow$": f satisfies (III) then $G := G_L$ can be developed into a power series (which has a radius of convergence 1 and therefore converges uniformly in $B_{\frac{1}{2}}(0)$ including 0!), hence $G(0) := a_0$ makes G continuous on the whole circle $B_1(0)$ with 0 included and by Riemann's theorem on removable singularities, this already makes G holomorphic. Given any sequence $z_n \to i\infty$, $\alpha(z_n) \to 0$ as the imaginary part controls $|e^{iz}|$ because

$$|e^{iz}| = \underbrace{|e^{i\operatorname{Re}(z)}|}_{=1} \quad \underbrace{|e^{-\operatorname{Im}(z)}|}_{\to 0 \text{ for } \operatorname{Im}(z)\to+\infty}$$

so $f_L(z_n) = G(\alpha(z_n)) \to G(0) = a_0$.
"$\Leftarrow$": Let c be a constant such that $f(i\infty) = c$, then we have to show that G is holomorphic at 0. By Riemann's theorem on removable singularities, it suffices to show that G is continuous at 0 so let $(t_n)_{n\in\mathbb{N}} \subset B_1(0)$ be a sequence converging to 0.

$$\begin{aligned} G(t_n) &= f_L(\beta(t_n)) \\ &= f_L\left(\frac{n_L \operatorname{Log}(t_n)}{2\pi i}\right) \\ &= f_L \underbrace{\left(i\frac{-n_L \log_{\mathbb{R}} |t_n|}{2\pi} + \frac{\operatorname{ph}(t_n)}{2\pi}\right)}_{:=z_n} \end{aligned} \tag{2.6}$$

We have

$$\operatorname{Im}(z_n) = \frac{n_L}{2\pi} \cdot \underbrace{(-\log_{\mathbb{R}} |t_n|)}_{\to+\infty} \to +\infty$$

because as $t_n \to 0$, so does $|t_n|$ so that $\log_{\mathbb{R}} |t_n| \to -\infty$ and therefore $-\log_{\mathbb{R}} |t_n| \to +\infty$. Consequently, $G(t_n) = f(z_n) \to f(i\infty) = c$ by the assumption and G is continuous and finally holomorphic in $B_1(0)$ with 0 included. Therefore, in a small neighborhood U around 0, G can be developed in a power series $G(t) = \sum_{n=0}^{\infty} b_n t^n$. Since power series are unique, the series from (2.4.4) on $B_1(0) \supset U$ must coincide with the one on U, i.e. $a_n = b_n$ for all $n \in \mathbb{N}$. Therefore

$$f_L(z) = G(\alpha(z)) = \sum_{n=-\infty}^{\infty} a_n q^n = \sum_{n=0}^{\infty} b_n q^n$$

and f satisfies condition (III). □

2.4.8 Corollary. *Let f be a function satisfying conditions (I) and (II) and let f_L be holomorphic at infinity in the sense that there exists a $c \in \mathbb{C}$ such that $f_L(i\infty) = c$, then c is the first term a_0 in the Laurent series of G_L, respectively in the Fourier expansion of f_L. In particular we can check for a zero at the cusp $L.\infty$ by computing the limit of f_L for $z \to i\infty$.*

Proof. $a_0 = \lim_{t\to 0} G_L(t) = \lim_{z\to i\infty} f_L(z) = c$ where the last equation comes from the proof of Theorem 2.4.7, formula (2.6). □

Notation: We will use $i\infty$ and ∞ in the same way, i.e. if we write $f(\infty)$ we mean $f(i\infty)$.

We now want to define the quantity called "order" (of f at some point $z_0 \in \overline{\mathbb{H}}$). For $z_0 \in \mathbb{H}$, we already have a term called order: as f is holomorphic in a neighborhood U around z_0 there exists a power series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \; \forall z \in U$. We define

$$\omega_f(z_0) := \inf\{0 \leq n \in \mathbb{N} \mid a_n \neq 0\}$$

A somewhat senseful continuation of $\omega_f(z_0)$ for a cusp $z_0 = L.\infty$ would be the first occurring term in some Laurent series that describes the behavior of f at $L.\infty$. Since f_L – in a certain sense – does this, we define:

2.4.9 Definition. *For Γ a subgroup of $SL_2(\mathbb{Z})$ having finite index, a cusp $L.\infty$ and the Fourier expansion $\sum_{n=0}^{\infty} a_n q_L^n$ of f_L guaranteed by condition (III), we define*

$$\omega_f(z_0) := \inf\{0 \leq n \in \mathbb{N} \mid a_n \neq 0\}$$

.

Again there is a hidden detail in this definition: As in the situation of $n_{z_0} = n_L$ (see 2.3.5) we write $\omega_f(z_0)$ but actually mean $\omega_f(L.\infty)$ i.e. again we give ω_f two things: the cusp and the way it got there from ∞. The question now again is whether this term is independent of the way and just depends on the cusp. The following will answer this positively but it is a 'theorem' as it will also answer a more central question afterwards:

2.4.10 Theorem. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having finite index, f a function that satisfies (I), (II) with weight $k \in \mathbb{Z}$ and a cusp $c = L.\infty = M.\infty$ for some $L, M \in SL_2(\mathbb{Z})$, then $f_L = \sum_{n\in\mathbb{Z}} a_n q_L^n$ and $f_M = \sum_{n\in\mathbb{Z}} b_n q_M^n$ exist by Theorem 2.4.4(b). As $n_L = n_M$ (see 2.3.5), $f_M = \sum_{n\in\mathbb{Z}} b_n q_L^n$, i.e. we can chose q_L instead of q_M here. The claim is that there exists $d \in \mathbb{Z}$ such that*

$$b_n = \epsilon^k \cdot a_n e^{\frac{2\pi i n d}{n_c}}$$

where $\epsilon \in \{+1, -1\}$ and consequently, as $\epsilon^k e^{2\pi i n d/n_c} \neq 0$, in particular

$$\inf\{0 \leq n \in \mathbb{N} \mid a_n \neq 0\} = \inf\{0 \leq n \in \mathbb{N} \mid b_n \neq 0\}.$$

Thus $\omega_f(z_0)$ is well-defined, i.e. $\omega_f(L.\infty) = \omega_f(M.\infty)$.

Proof. Since $L.\infty = c = M.\infty$, we have $L^{-1}M.\infty = \infty$ so by Theorem 2.2.3, there is a $d \in \mathbb{Z}$ such that $L^{-1}M = \epsilon U^d$ for some $\epsilon \in \{+1, -1\}$. Then the following relation holds:

$$
\begin{aligned}
\sum_{n\in\mathbb{Z}} a_n q_L^n &= G_M(e^{2\pi i z/n_c}) = G_M(q_M) \\
&= f_M(z) \\
&= \frac{1}{(M:z)^k} f(M.z) \\
&= \frac{1}{\epsilon^k (LU^d:z)^k} f(LU^d.z) && (+LU^d.z = -LU^d.z) \\
&= \frac{1}{\epsilon^k (L:U^d)^k (0z+1)^k} f(L.(z+d)) \\
&= \epsilon^k \frac{1}{(L:z+d)^k} f(L.(z+d)) \\
&= \epsilon^k f_L(z+d) \\
&= \epsilon^k G_L(q_L e^{2\pi i d/n_c}) \\
&= \sum_{n\in\mathbb{Z}} (\epsilon^k e^{\frac{2\pi i n d}{n_c}} a_n) q_L^n
\end{aligned}
$$

The uniqueness of Laurent series (applied to the functions $t \in B_1(0) \setminus \{0\} \mapsto \sum_{n\in\mathbb{Z}} (\epsilon^k e^{\frac{2\pi i n d}{n_c}} a_n) t^n$ and $t \in B_1(0) \setminus \{0\} \mapsto \sum_{n\in\mathbb{Z}} a_n t^n$ which coincide on $B_1(0) \setminus \{0\}$ according to the above) now implies that these series have to coincide component wise which is the assertion claimed. □

This theorem also gives another major insight into condition (III). Instead of verifying (III) for every $L \in \mathrm{SL}_2(\mathbb{Z})$, it suffices to verify this condition at each cusp, i.e. (III) is not a restriction of the "general behavior" of f but rather only a restriction of the "behavior near each cusp":

2.4.11 Theorem. *Let Γ be a subgroup having finite index in $SL_2(\mathbb{Z})$, f a function satisfying (I), (II) and let $z_1 = L_1.\infty, ..., z_n = L_n.\infty \in \mathbb{P}$ be a system of representatives for the cusp orbits of Γ, i.e. $\mathrm{cu}(\Gamma) = \{[\![z_1]\!], ..., [\![z_n]\!]\}$ and let f satisfy (III) for $L_1, ..., L_n$, then f satisfies condition (III) for all $L \in SL_2(\mathbb{Z})$.*

Proof. Let $L \in \mathrm{SL}_2(\mathbb{Z})$ and since $L.\infty \in \mathrm{cu}(\Gamma)$, there is an m with $L.\infty = L_m.\infty$. Using Theorem 2.4.10, we see that the Laurent expansions of G_L and G_{L_m} are the same up to multiplication of a nonzero constant, i.e. since G_{L_m} starts with a term carrying a positive index, so does G_L. □

2.5 Examples of modular forms

We will now construct modular forms of weight 4 for $\Gamma_0(2)$ and weight 3 for $\Gamma_1(3)$ respectively. We need the following preparatory results:

2.5.1 Proposition. *If some function f of weight $k \in \mathbb{Z}$ transforms correctly under matrices $X, Y \in \Gamma$, then it also transforms correctly under $X \cdot Y$.*

Proof.

$$f(X \cdot Y.z) = f(X.Y.z) = (X : Y.z)^k f(Y.z)$$
$$= [(X : Y.z)(Y : z)]^k f(z) \stackrel{2.4.2}{=} (XY : z)^k f(z)$$

. □

The assertion of this proposition is that in order to verify condition (II), we may only verify this condition for generators of the subgroup. Therefore we need to see that $\Gamma_1(3)$ and $\Gamma_0(2)$ are generated by a few – hopefully relatively simple to handle – matrices. This is indeed the case:

2.5.2 Lemma. *(a)* $\Gamma_0(2)$ *as a group is generated by the matrices*

$$-Id = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ VU^2V^{-1} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$$

(b) $\Gamma_1(3)$ *as a group is generated by the matrices*

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ VU^3V^{-1} = \begin{pmatrix} 1 & 0 \\ -3 & 1 \end{pmatrix}$$

Proof. (b): First we need two intermediate results:

$$\forall x, y \in \mathbb{Z}, y \neq 0 \Rightarrow \exists m \in \mathbb{Z} \ |x + my| < |y| \qquad (*)$$

Since $y \neq 0$ so is $|y|$ so that, using the euclidean algorithm, we can write $x = k|y| + r$ where $k \in \mathbb{Z}, r \in \mathbb{N}, 0 \leq r < |y|$ and hence we have

$$|x - k|y|| = x - k|y| \qquad \text{as } x - k|y| = r \text{ and } r \geq 0$$
$$= r < |y|$$

So if $y < 0$ then $|y| = -y$ and $m = +k$ will do, otherwise we select $m = -k$.

$$\forall x, y \in \mathbb{Z}, y \neq 0 \Rightarrow \exists m \in \mathbb{Z} \ |x + my| < \frac{2}{3}|y| \qquad (**)$$

Using (*), we obtain $m \in \mathbb{Z}$ so that $|x+my| < |y|$. If $|x+my| < 2/3|y|$ already holds then we are done. Otherwise we have $|y| > |x+my| \geq 2/3|y|$ and distinguish the following cases:

1st case: $x + my \leq 0$, then we have

$$\begin{aligned} & -|y| < x + my \leq -\frac{2}{3}|y| \\ \Rightarrow & 0 = -|y| + |y| < x + my + |y| \leq -\frac{2}{3}|y| + |y| = \frac{1}{3}|y| \\ \Rightarrow & |x + my + |y|| = x + my + |y| \leq \frac{1}{3}|y| < \frac{2}{3}|y| \end{aligned}$$

So either $m+1$ or $m-1$, depending on whether $y < 0$ or $y > 0$ will do.

2nd case: $x + my \geq 0$, then we have

$$\begin{aligned} & \frac{2}{3}|y| \leq x + my < |y| \\ \Rightarrow & -\frac{1}{3}|y| = \frac{2}{3}|y| - |y| \leq x + my - |y| < |y| - |y| = 0 \\ \Rightarrow & |x + my + |y|| = -(x + my + |y|) \leq \frac{1}{3}|y| < \frac{2}{3}|y| \end{aligned}$$

So either $m+1$ or $m-1$, depending on whether $y < 0$ or $y > 0$ will do.

Now we begin the main part of the proof: Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be in $\Gamma_1(3)$. We will then iteratively construct a sequence

$$M^{(1)} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, M^{(2)} = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, \ldots$$

such that

$$|c| > |c_1| > |c_2| > \ldots$$

so the $|c_i|$ will form a strictly decreasing chain in $\mathbb{N}$ which – after finitely many steps – has to stagnate at $|c_N|$. We will only show how to get from M to M^1 as the rest is done inductively. First we see that

$$U^{\pm m} M = \begin{pmatrix} a + mc & * \\ c & * \end{pmatrix}$$

and

$$(VU^3V^{-1})^{\pm m} M = (VU^{\pm 3m}V^{-1})M = \begin{pmatrix} a & * \\ c \mp 3ma & * \end{pmatrix}$$

i.e. using U, we can move a in step width of c without touching c and using VU^3V^{-1}, we can move c with a step width of $3a$ without touching c. Using

(**), applied to a, c, we obtain $m \in \mathbb{Z}$ s.t. $|a + mc| < 2/3|c|$. Thus, for the matrix

$$M' = U^m M = \begin{pmatrix} a' & * \\ c & * \end{pmatrix}$$

where $a' = a + mc$ and $|a'| < 2/3|c|$. the crucial step now is that we can decrease the size of c using a': We claim that there is an $m' \in \mathbb{Z}$ with $|c + 3m'a| < |c|$. To prove this we distinguish the following cases:

<u>1st case</u>: $a' > 0, c > 0$. Then

$$|c - 3a'| < |c| = c$$
$$\Updownarrow$$
$$c - 3a' < c \iff -3a' < 0 \iff a' \geq 0 \checkmark \text{ (by assumption)}$$
and
$$-c + 3a' < c \iff 3a' < 2c \iff a' < \frac{2}{3}c$$

and the latter one is true because $a' = |a'|, c = |c|$ as both are positive and $|a'| < 2/3|c|$ by (**).

<u>2nd case</u>: $a' < 0, c > 0$. Then

$$|c + 3a'| < |c| = c$$
$$\Updownarrow$$
$$c + 3a' < c \iff 3a' < 0 \iff a' < 0 \checkmark \text{ (by assumption)}$$
and
$$-c - 3a' < c \iff -3a' < 2c \iff -a' < \frac{2}{3}c$$

and the latter one is true because $-a' = |a'|, c = |c|$ and $|a'| < 2/3|c|$ by (**).

<u>3rd case</u>: $a' > 0, c < 0$. Then

$$|c + 3a'| < |c| = -c$$
$$\Updownarrow$$
$$c + 3a' < -c \iff 3a' < -2c \iff a' < \frac{2}{3}(-c)$$
and
$$-c - 3a' < -c \iff -3a' < 0 \iff a' > 0 \checkmark \text{ (by assumption)}$$

and the first one is true because $a' = |a'|, (-c) = |c|$ and $|a'| < 2/3|c|$ by (**).

<u>4th case</u>: $a' < 0, c < 0$.

$$|c - 3a'| < |c| = -c$$
$$\Updownarrow$$
$$c - 3a' < -c \iff -3a' < -2c \iff -a' < \frac{2}{3}(-c)$$
$$\text{and}$$
$$-c + 3a' < -c \iff 3a' < 0 \iff a' < 0 \checkmark \text{ (by assumption)}$$

and the first one is true because $-a' = |a'|, (-c) = |c|$ and $|a'| < 2/3|c|$ by (**). In any case $m' = +1$ or $m' = -1$ will fit our needs. All in all we now have constructed the matrix

$$M^{(1)} = VT^{\pm 3}V^{-1} \cdot U^m \cdot M = \begin{pmatrix} a' & * \\ c + m'a' & * \end{pmatrix}$$

where $|c_1| = |c + m'a'| < |c|$.

Note that the case $a = 0$ can never occur as $M \in \Gamma_1(3)$ so $a \equiv 1 \mod 3$ but $a = 0$ implies $a \equiv 0$. Also note that the case $c = 0$ may occur but then we are directly in the situation as if the chain had stagnated and 'start' the proof below.

Restarting this process with $M^{(1)}$ yields $M^{(2)}$ and so forth. When the chain stagnates we have found integers $m_1, m_1', m_2, m_2', ...$ such that

$$VU^{3m_N'}V^{-1}U^{m_N} \cdots VU^{3m_1'}V^{-1}U^{m_1}M = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = W, \tag{2.7}$$

Where the c-entry in W is zero as otherwise we could apply the above once more and find a c_{N+1} with $|c_{N+1}| < |c_N|$ i.e. the chain had not stagnated yet which is a contradiction. It is possible that $N = 0$. In that case c was 0 from the beginning.

As $W \in \mathrm{SL}_2(\mathbb{Z})$, $|x| = |y| = 1$ and x, y share the same sign. Actually they share $+$ as $x = -1$ would imply that $x \equiv 2 \not\equiv 1 \mod 3$ which is a contradiction because $W \in \Gamma_1(3)$ as all matrices in (2.7) are. So $W = \left(\begin{smallmatrix} 1 & r \\ 0 & 1 \end{smallmatrix}\right) = U^r$ for some $r \in \mathbb{Z}$ but then we have shown that

$$M = U^{-3m_1}VU^{-m_1'}V^{-1} \cdots U^{-3m_N}VU^{-m_N'}V^{-1}U^r$$

is in the group generated by U and VU^3V^{-1} as desired.

On (a): The proof is actually the same. One needs to see that (**) works with the factor 1/2 too and finally one ends up with W as above. One then needs $-Id$ because the case $x = -1$ can not be excluded. □

2.5.3 Definition. *We define the Dedekind η function to be*

$$\eta(z) = e^{\frac{2\pi i z}{24}} \cdot \prod_{n\in\mathbb{N}} (1 - e^{2\pi i n z}) = q^{1/24} \prod_{n\in\mathbb{N}} (1 - q^n).$$

This function is not itself a modular form but due to the following two facts it can be used to construct modular forms for subgroups:

2.5.4 Theorem. *η is well-defined and converges uniformly on sets of the form*

$$S_\delta = \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq \delta > 0\}$$

for any $\delta \in \mathbb{R}^+$.

Proof. See [RS 07, Ko 93]. □

2.5.5 Theorem. *η satisfies the following transformation rule:*

$$\eta(-\frac{1}{z}) = \sqrt{\frac{z}{i}}\eta(z)$$

where $\sqrt{z} = e^{1/2\,\mathrm{Log}(z)}$ is the branch of the complex root having positive real part.

Proof. See [Ko 93]. □

2.5.6 Conclusions. *η is in particular locally uniformly convergent and as a locally uniformly convergent limit of holomorphic functions it is holomorphic (by the theorem of Weierstrass). Furthermore, η is holomorphic at $i\infty$: With the use of Cauchy sequences and the uniformly convergence on the sets S_δ, one can show that the well known rule from analysis,*

$$\lim_{z\to z_0} \lim_{n\to\infty} f_n(z) = \lim_{n\to\infty} \lim_{z\to z_0} f_n(z)$$

extends to the point $z = i\infty$ Hence:

$$\begin{aligned} \lim_{z\to i\infty} \prod_{n\in\mathbb{N}} (1 - q^n) &= \prod_{n\in\mathbb{N}} (1 - \lim_{z\to i\infty} q^n) \\ &= 1 \end{aligned}$$

With the help of Lemma 2.5.2 and the above transformation rule we can show that

2.5.7 Theorem. *(a) $\varphi(z) := \frac{\eta(2z)^{16}}{\eta(z)^8}$ is a modular form for $\Gamma_0(2)$ of weight 4. It vanishes at the cusp ∞ and does not vanish at the cusp 0.*

(b) $\psi(z) := \frac{\eta(3z)^9}{\eta(z)^3}$ *is a modular form for* $\Gamma_1(3)$ *of weight* 3*. It vanishes at the cusp* ∞ *and does not vanish at the cusp* 0*.*

Proof. We need to show that these functions satisfy the three conditions. Both satisfy (I) because $\eta(z) \neq 0$ for all $z \in \mathbb{H}$ and by the conclusions above, η itself is holomorphic. Thus, in both cases the function is holomorphic as a composition of such. By 2.5.1 we only need to verify (II) for the generators which we have computed in Lemma 2.5.2. We will only do the following computations for φ as the proof for ψ is actually completely the same with different numbers substituted and is therefore left to the reader.

On (a): Set $\varphi(z) = \frac{\eta(2z)^{16}}{\eta(z)^8}$.

(II):

$$\begin{aligned}
\varphi(Uz) &= \varphi(z+1) \\
&= \frac{\eta(2(z+1))^{16}}{\eta(z+1)^8} \\
&= \frac{(e^{2\pi i 2/24})^{16}\ [e^{2\pi i 2z/24} \prod_{n\in\mathbb{N}} (1 - \cancel{e^{2\pi i n}} e^{2\pi i n z})]^{16}}{(e^{2\pi i 1/24})^{8}\ [e^{2\pi i z/24} \prod_{n\in\mathbb{N}} (1 - \cancel{e^{2\pi i n}} e^{2\pi i n z})]^{8}} \\
&= e^{2\pi i (32-8)/24} \varphi(z) \\
&= \varphi(z)
\end{aligned}$$

and for VU^2V^{-1} we first analyze nominator and denominator:

$$\begin{aligned}
\eta(2\cdot VU^2V^{-1}.z)^{16} &= \eta\left(\frac{2z}{-2z+1}\right)^{16}\\
&= \eta\left(\frac{1}{\frac{1}{2z}(-2z+1)}\right)^{16}\\
&= \eta\left(\frac{1}{-1+\frac{1}{2z}}\right)^{16}\\
&= \eta\left(\frac{1}{\frac{1}{2z}-1}\right)^{16}\\
&= \eta\left(-\frac{1}{1-\frac{1}{2z}}\right)^{16}\\
&= \left(\sqrt{\left(1-\frac{1}{2z}\right)\left(\frac{1}{i}\right)}\right)^{16}\eta\left(1-\frac{1}{2z}\right)^{16} \quad \text{(by (2.5.5))}\\
&= \left(1-\frac{1}{2z}\right)^{8}\underbrace{(-i)^8}_{=1}\eta\left(1-\frac{1}{2z}\right)^{16}
\end{aligned}$$

for the denominator we obtain

$$\begin{aligned}
\eta(VU^2V^{-1}.z)^8 &= \eta\left(\frac{z}{-2z+1}\right)^8 \\
&= \eta\left(\frac{1}{\frac{1}{z}(-2z+1)}\right)^8 \\
&= \eta\left(\frac{1}{-2+\frac{1}{z}}\right)^8 \\
&= \eta\left(\frac{1}{\frac{1}{z}-2}\right)^8 \\
&= \eta\left(-\frac{1}{2-\frac{1}{z}}\right)^8 \\
&= \left(\sqrt{\left(2-\frac{1}{z}\right)\left(\frac{1}{i}\right)}\right)^8 \eta\left(2-\frac{1}{z}\right)^8 \qquad \text{(by (2.5.5))} \\
&= \left(2-\frac{1}{z}\right)^4 \underbrace{(-i)^4}_{=1} \eta\left(2-\frac{1}{z}\right)^8
\end{aligned}$$

All in all we obtain for φ:

$$\varphi(VU^2V^{-1}.z) = \frac{\left(1-\frac{1}{2z}\right)^8 (e^{2\pi i/24})^{16}\left(e^{\frac{2\pi i\left(-\frac{1}{2z}\right)}{24}}\prod_{n\in\mathbb{N}}(1-\cancel{e^{2\pi in(1)}}e^{2\pi in\left(-\frac{1}{2z}\right)})\right)^{16}}{\left(2-\frac{1}{z}\right)^4 (e^{2\pi i2/24})^{8}\left(e^{\frac{2\pi i\left(-\frac{1}{z}\right)}{24}}\prod_{n\in\mathbb{N}}(1-\cancel{e^{2\pi in(2)}}e^{2\pi in\left(-\frac{1}{z}\right)})\right)^{8}}$$

$$= e^{\frac{2\pi i}{24}(16-2\cdot 8)}\ \frac{\left(1-\frac{1}{2z}\right)^8}{\left(1-\frac{1}{z}\right)^4}\ \frac{\eta\left(-\frac{1}{2z}\right)^{16}}{\eta\left(-\frac{1}{z}\right)^{8}}$$

$$= \frac{\left(1-\frac{1}{2z}\right)^8}{\left(2-\frac{1}{z}\right)^4}\frac{\sqrt{2z/i}^{16}\eta(2z)^{16}}{\sqrt{z/i}^{8}\eta(z)^{8}} \qquad \text{(by (2.5.5))}$$

$$= \frac{\left(2z-\frac{2z}{2z}\right)^8}{\left(z-\frac{z}{z}\right)^4}\frac{\cancel{(-i)^8}}{\cancel{(-i)^4}}\varphi(z)$$

$$= \frac{(2z-1)^8}{(2z-1)^4}\varphi(z)$$

$$= (-1)^4(2z-1)^4\varphi(z)$$

$$= (-2z+1)^4\varphi(z)$$

$$= (VU^2V^{-1} : z)^4\varphi(z)$$

Under $-Id$, φ transforms correct as $(-Id : z)^4 = (-1)^4 = 1$ and $-Id.z = Id.z = z$.

(III): Note that we only need to verify (III) at the cusps according to 2.4.11. φ vanishes at the cusp ∞:

$$\lim_{z\to i\infty}\varphi(z) = \lim_{z\to i\infty} e^{2\pi iz}\cdot\frac{\left(\prod_{n\in\mathbb{N}}(1-\lim_{z\to i\infty}e^{2\pi in2z})\right)^{16}}{\left(\prod_{n\in\mathbb{N}}(1-\lim_{z\to i\infty}e^{2\pi inz})\right)^{8}}$$

$$= 0\cdot\frac{\prod 1}{\prod 1} = 0$$

where we were able to switch limit and product as in 2.5.6 explained.

φ is holomorphic and does not vanish at the cusp 0:

$$\begin{aligned}\varphi(V.z) &= \frac{\eta\,(2\cdot Vz)^{16}}{\eta\,(Vz)^{8}} \\ &= \frac{\eta\left(-\frac{1}{\frac{z}{2}}\right)^{16}}{\eta\left(-\frac{1}{z}\right)^{8}} \\ &= \frac{\left(\sqrt{\left(\frac{z}{2}\right)\left(\frac{1}{i}\right)}\right)^{16}\eta\left(\frac{z}{2}\right)^{16}}{\left(\sqrt{z\left(\frac{1}{i}\right)}\right)^{8}\eta\,(z)^{8}} && \text{(by (2.5.5))} \\ &= \frac{1}{2^8}\frac{\cancel{z^8}^{\,4}}{\cancel{z^4}}\underbrace{(-i)^8}_{=1}\underbrace{(-i)^{-4}}_{=1}\underbrace{e^{\frac{2\pi iz}{24}(16/2-8)}}_{=e^0=1}\frac{\prod\left(1-q^{n/2}\right)}{\prod\left(1-q^n\right)}\end{aligned}$$

so for $\varphi_V(z)$ we obtain

$$\begin{aligned}\varphi_V(z) &= \frac{1}{(V:z)^4}\varphi(Vz) \\ &= \frac{\cancel{1}}{\cancel{z^4}}\cancel{z^4}\frac{1}{2^8}\frac{\overbrace{\prod\left(1-q^{n/2}\right)}^{\to 1}}{\underbrace{\prod\left(1-q^n\right)}_{\to 1}} \\ &\xrightarrow{z\to i\infty} \frac{1}{2^8}\cdot 1\end{aligned}$$

where again the convergence of the products follows from 2.5.6.

□

For eventually proving the main result in section 4 we need two more modular forms.

2.5.8 Definition. *We define the lattices*

- $A_2 = \{(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_1,x_2,x_3\in\mathbb{Z}, x_1+x_2+x_3=0\}$
  $= \langle(1,-1,0),(0,1,-1)\rangle_{\mathbb{Z}}$
- $D_4 = \langle(1,-1,0,0),(0,1,-1,0),(0,0,1,-1),(0,0,1,1)\rangle_{\mathbb{Z}}$

These are the root lattices of the respective Lie algebras. The Theta-series of some lattice L is defined as the formal sum

$$\Theta_L(z) := \sum_{l\in L} e^{\pi i(l|l)}$$

where $(\cdot|\cdot)$ *is the usual euclidean scalar product on* $\mathbb{R}^n$*.*

One can show that this series converges locally uniformly and that the following holds:

2.5.9 Theorem. *Consider* D_4 *and* A_2*, the root lattices for the respective Lie algebras. Then the Theta-series corresponding to these lattices are modular forms:*

(a) $\Theta_2(z) := \Theta_{D_4}(z)$ *is a modular form for* $\Gamma_0(2)$ *of weight* 2*.*

(b) $\Theta_1(z) := \Theta_{A_2}(z)$ *is a modular form for* $\Gamma_1(3)$ *of weight* 1*.*

Proof. We use [Eb 00], Thm. 3.2, p. 99. Theta series generally do neither transform correct under all $S \in \mathrm{SL}_2(\mathbb{Z})$ nor under all $S \in \Gamma$ but in this case we have for Θ_2: $\det(D_4) = 2^2 = 4$, $n = 4$, $\Delta = 2^2$ and $\left(\frac{\Delta}{d}\right) = \pm 1$ but as the Dirichlet symbol is multiplicative and Δ is a square, $\left(\frac{\Delta}{d}\right) = +1$ so its character is trivial. For Θ_1 we have $\det(A_2) = 3, n = 2, \Delta = -3$ and for any matrix $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(3)$, we have $d \equiv 1 \mod 3$ and since the Dirichlet symbol is compatible with this, $\left(\frac{\Delta}{d}\right) = \left(\frac{-3}{1}\right) = 1$ by definition of the symbol in this case, so Θ_1 transforms correctly for all matrices in $\Gamma_1(3)$. □

3 The $k/12$ – Formula

Modular forms are often given as infinite products or sums so there is no chance for finding zeros in a direct computational approach. Unfortunately, finding zeros is exactly what we need to do in order to show that the modular forms which will generate $M(\Gamma_0(2))$ – $M(\Gamma_1(3))$ respectively – are algebraically independent. Therefore one needs to extract the information whether or whether not $f(z_0) = 0$ differently. For $z_0 \in \mathbb{H}$, one possibility is given by measuring $\omega_f(z_0)$. If $N := \omega_f(z_0) > 0$ then the Fourier series (which is actually a power series as f is holomorphic) begins with a term having positive index, so $f(z) = \sum_{n=N}^{\infty} a_n(z - z_0)^n = (z - z_0)^N \cdot h(z)$, h being holomorphic and therefore $f(z_0) = (z_0 - z_0)^N \cdot h(z_0) = 0$. It is however very difficult to measure $w_f(z_0)$ for some specific $z_0 \in \mathbb{H}$ directly. As we will see: a better way is to sum up all information on all possible zeros (including the possible zeros at the cusps), called the *total order* of f, and then measure this term. This has two reasons: firstly, this term is equal to a very nice expression. Secondly, one has put a lot of effort on studying the Fourier series of the functions f_L (mostly the coefficients of these expansions are the objects of desire) so using Corollary 2.4.8, one has information about the behavior of f_L at infinity. Substituting the information of the behavior at the cusps into the formula first mentioned will relate the zeros at the cusps and actual zeros in $\mathbb{H}$ in a very nice way. Thus, the subject is now to show this equality.

A consequence of the generalized definition of $\omega_f(z)$ is that the order does not depend on the concrete representative z but only on the orbit $\Gamma.z = [\![z]\!]$, i.e. $\omega_f([\![z]\!]) := \omega_f(z)$ is well-defined. Before we prove this we need a preparatory lemma:

3.0.1 Lemma. *Let G be an area, $z_0 \in G$ fixed, $f : G \to \mathbb{C}$ a meromorphic function unequal to the zero function and $h : G \to G$ a holomorphic bijective function such that $h'(z_0) \neq 0$ and h^{-1} is holomorphic again, then we have $w_f(h(z_0)) = w_{f\circ h}(z_0)$, i.e. the behavior of f at $h(z_0)$ is the behavior of $f \circ h$ at z_0.*

Proof. From complex analysis we know that $\omega_f(h(z_0)) = N$ if and only if there is a neighborhood U around z_0 such that the function $r : U \to \mathbb{C}$ with $r(z) = (z - h(z_0))^{-N} f(z)$ is holomorphic on U with $r(h(z_0)) \neq 0$. Analogously, $\omega_{f\circ h}(z_0) = M$ iff. $s(v) = (v - z_0)^{-M} f(h(v))$ is holomorphic at z_0 with $s(z_0) \neq 0$. We show $N = M$ by showing that s with exponent N is holomorphic at z_0 with $(v-z_0)^{-N} f(h(z_0)) \neq 0$. Let a sequence $(v_n)_{n\in\mathbb{N}}$ with $v_n \overset{n\to\infty}{\longrightarrow} z_0$ be given. Set $z_n := h(v_n)$. Since h is in particular continuous, we have $z_n \overset{n\to\infty}{\to} h(z_0)$. Consider the quotient $r(z_n)/s(v_n)$:

$$\begin{aligned}
\frac{r(z_n)}{s(v_n)} &= \frac{(z_n - h(z_0))^{-N} f(z_n)}{(v_n - z_0)^{-N} f(h(v_n))} \\
&= \frac{(h(v_n) - h(z_0))^{-N} \cancel{f(h(v_n))}}{(v_n - z_0)^{-N} \cancel{f(h(v_n))}} && (\text{as } z_n = h(v_n)) \\
&= \left(\frac{h(v_n) - h(z_0)}{v_n - z_0} \right)^{-N} \\
&\overset{n\to\infty}{\longrightarrow} (h'(z_0))^{-N} \neq 0 && (3.1)
\end{aligned}$$

The cancellation makes sense as $f(h(v_n)) \neq 0$ for all but finitely many $n \in \mathbb{N}$ because either f is holomorphic at $h(z_0)$ with $f(h(z_0)) \neq 0$ (then $N = M = 0$) or $h(z_0)$ is a pole of f (then there must be a neighborhood U' such that $|f(z')| > 1 \ \forall z' \in U'$. In the case that $h(z_0)$ is a zero of f the zeros cannot accumulate as f is holomorphic and not the zero function. Moreover, $h'(z_0)^{-N}$ is defined as $h'(z_0) \neq 0$ by assumption. We have shown that s is continuous in z_0. By Riemann's theorem on removable singularities, this already suffices to show that s is holomorphic in z_0. Its value at z_0 is computed to be

$$s(z_0) = \lim_{v \to z_0} s(v) \overset{(3.1)}{=} \underbrace{r(h(z_0))}_{\neq 0} \underbrace{h'(z_0)^N}_{\neq 0} \neq 0$$

Hence $N = M$. $\square$

3.0.2 Theorem. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$, then*

(i) $\forall z_1, z_2 \in \mathbb{H}, z_1 \approx z_2 \mod \Gamma \Rightarrow w_f(z_1) = w_f(z_2)$

(ii) $\forall L_1, L_2 \in SL_2(\mathbb{Z}), L_1 \sim L_2 \mod \Gamma \Rightarrow n_{L_1} = n_{L_2}$

(iii) $\forall L_1, L_2 \in SL_2(\mathbb{Z}), L_1 \sim L_2 \mod \Gamma \Rightarrow \widehat{n_{L_1}} = \widehat{n_{L_2}}$

(iv) The first rule also holds for cusps, i.e. $L.\infty \approx L'.\infty \mod \Gamma$ *then* $\omega_f(L.\infty) = \omega_f(L'.\infty)$.

In short: order and width of points and cusps do not depend on the concrete representative but rather on the orbit. As a reformulation, we obtain the cancellation rules $\omega_f(T.z) = \omega_f(z)$, $n_{TL} = n_L$ *and* $\widehat{n_{TL}} = \widehat{n_L}$ *for all* $z \in \overline{\mathbb{H}}, T \in \Gamma, L \in SL_2(\mathbb{Z})$. *Consequently,* $\zeta = [\![L.\infty]\!] \in \mathrm{tr}(\infty)$, *then* $\omega_f(\zeta) := \omega_f(L.\infty)$, $n_\zeta := n_L$ *and* $\widehat{n_\zeta} := \widehat{n_L}$ *are well-defined.*

Proof. (i): Let $z_2 = T.z_1$ with $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Set the function $h : \mathbb{H} \to \mathbb{H}$ to be $h(z) = T.z$. Then, $h'(z) \neq 0$ for all $z \in \mathbb{H}$ as

$$\begin{aligned} h'(z) &= \left(\frac{az+b}{cz+d}\right)' \\ &= \frac{a(cz+d) - (az+b)c}{(cz+d)^2} \\ &= \frac{\cancel{acz} + ad - \cancel{acz} - bc}{(cz+d)^2} \\ &= \frac{\overbrace{\det(T)}^{=1}}{(cz+d)^2} \neq 0 \end{aligned}$$

because $\mathrm{Im}(z) > 0$ and c, d are real. Hence we can apply the above lemma to f and h and obtain for every $z_0 \in \mathbb{H}$:

$$\begin{aligned}\omega_f(T.z_0) &= \omega_f(h(z_0)) \\ &= \omega_{(f\circ h)}(z_0) && \text{(by Lemma 3.0.1)} \\ &= \omega_{(cz+d)^k \cdot f}(z_0) && (f \text{ satisfies (II)}) \\ &= \underbrace{\omega_{(cz+d)^k}(z_0)}_{=0} + \omega_f(z_0) && \text{(complex analysis)} \\ &= \omega_f(z_0)\end{aligned}$$

Remark that the usage of holomorphicity and the above lemma makes a separate proof for the cusps necessary.

(ii): $L_1 \sim L_2 \mod \Gamma \Rightarrow (\exists\, T \in \Gamma)\ L_2 = TL_1$. Let $\widehat{S}$ be the generator of $\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L_1.\infty}$. By Lemma 2.2.5,

$$\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L_2.\infty} = \widehat{\mathrm{SL}_2(\mathbb{Z})}_{T.L_1.\infty} = \Phi(T)\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L_1.\infty}\Phi(T)^{-1} = \langle \widehat{T}\widehat{S}\widehat{T}^{-1} \rangle,$$

so the generator of the new stabilizer is $\widehat{S}\,' = \Phi(TST^{-1})$. To show $n_{L_1} = n_{L_2}$ it suffices to show that $\widehat{S} \in \widehat{\Gamma} \iff \widehat{S}\,' \in \widehat{\Gamma}$ since then the minimum is captured over exactly the same subset of exponents. The iff. holds as $T \in \Gamma$ implies $\widehat{T} = \Phi(T) \in \widehat{\Gamma}$ so

$$\widehat{S} \in \widehat{\Gamma} \iff \widehat{T}\widehat{S} \in \widehat{\Gamma} \iff \widehat{T}\widehat{S}\widehat{T}^{-1} = \widehat{S}\,' \in \widehat{\Gamma}$$

.

(iii): As $\widehat{n_L} = n_L/c$ (for $c \in \{1,2\}$ and $c = 2 \iff -Id \notin \Gamma$) for all $L \in \mathrm{SL}_2(\mathbb{Z})$ by 2.3.8,

$$\widehat{n_{L_1}} = n_{L_1}/c \overset{(ii)}{=} n_{L_2}/c = \widehat{n_{L_2}}$$

(iv): Let $L.\infty \approx L'.\infty$ both be in ζ, then there is a $G \in \Gamma$ such that $G.L.\infty = L'.\infty$. We have to show that $\omega_f(L.\infty) = \omega_f(L'.\infty)$. As ω_f does not depend on the way (see 2.4.10), we know that $\omega_f(L'.\infty) = \omega_f(G.L.\infty)$ and we only have to show

$$\omega_f(G.L.\infty) = \omega_f(L.\infty)$$

For every modular form f of weight k we obtain

$$\begin{aligned}f_{GL}(z) &= \frac{1}{(G:Lz)^k(L:z)^k}(G:Lz)^k \cdot f(L.z) \quad (G \in \Gamma, (II) \text{ and } 2.4.2) \\ &= f_L(z)\end{aligned}$$

As $n_L = n_{GL}$, the Fourier series of both function are completely the same and in particular their first occurring exponent is the same. □

3.1 The $k/12-$ Formula for modular forms for $\mathrm{SL}_2(\mathbb{Z})$

3.1.1 Theorem. *Let f be a modular form for $SL_2(\mathbb{Z})$ of weight k unequal to the zero form, then the following identity holds:*

$$\omega_f(\infty) + \frac{1}{2}\omega_f(i) + \frac{1}{3}\omega_f(\varrho) + \sum_{\substack{\zeta \in \mathbb{H} \cap \mathbb{F}_I \\ \zeta \not\approx i,\varrho \mod SL_2(\mathbb{Z})}} \omega_f(\zeta) = \frac{k}{12}$$

Proof. See [Ra 77], p. 98, Theorem 4.1.4. □

Remark that the sum $\sum_{\zeta \not\approx i,\varrho} \omega_f(\zeta)$ is finite: Assume there were infinitely many zeros of f in $\mathbb{F}_I$ (say $z_1, z_2, ...$ sorted in the sense that $\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2) \leq ...$). Using α on all of them, we obtain a sequence $t_n = \alpha(z_n)$ (α behaves injective because $\mathrm{Re}(z_n) \in [-1/2, 1/2)$). Hence, in the compact set $\overline{B_1(0)}$, there must be an accumulation point. Since the z_n are ordered in their imaginary part, $|t_1| \geq |t_2| \geq ...$ so the accumulation point t has to lie in $B_{|t_1|}(0) \subset B_1(0)$. Consider the associated function from condition (III), $G_{Id} = \sum_{n=0}^{\infty} a_n t^n$ for the cusp $Id.\infty$. As $G(\alpha(z)) = f(z)$, for a subsequence of $t_1, t_2, ...$ converging against t, we obtain an accumulation point of zeros of G as

$$G(t_n) = G(\alpha(z_n)) = f(z_n) = 0.$$

Since G_{Id} is holomorphic, G_{Id} is identically and since Laurent series are unique, $a_n = 0$ for all $n \in \mathbb{N}$. Thus,

$$f = f_{Id} = \sum a_n e^{2\pi i n z} = \sum 0 e^{2\pi i n z} = 0$$

is the zero form in contradiction to the assumption.
The term on the left in Theorem 3.1.1 is also called the "total order" of f, for short: $\mathrm{tot}(f)$.

3.2 The $k/12-$ Formula for modular forms for general subgroups

We now want to derive a formula for subgroups of finite index similar to the $k/12-$Formula. The tactic to do so is the following: Given a modular form f for some Γ of weight k, we can construct a modular form g for $\mathrm{SL}_2(\mathbb{Z})$ having a weight $k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]$. We then know a relation for the total order of g, namely $\mathrm{tot}(g) = k[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]/12$. The main question then is: how to define the "total order" for a modular form for a subgroup of $\mathrm{SL}_2(\mathbb{Z})$? How does the relation of the order of g and the order of f look like? These questions will be answered after the construction of g.

3.2.1 Theorem. *Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$, f be a modular form of integer weight $k \in \mathbb{Z}$ for Γ and $\mathcal{R}$ a RRS for $SL_2(\mathbb{Z})/\Gamma$. Set*

$$g(z) := \prod_{L \in \mathcal{R}} f_L(z)$$

then g is a modular form for $SL_2(\mathbb{Z})$ of weight $k[SL_2(\mathbb{Z}) : \Gamma]$.

Before we prove this theorem we need a lemma:

3.2.2 Lemma. *Let $\mathcal{R} = \{L_1, ..., L_n\}$ be a RRS for $SL_2(\mathbb{Z})/\Gamma$ and $X \in SL_2(\mathbb{Z})$ fixed. For every $L \in \mathcal{R}$, there are unique matrices S_L, R_L such that $S_L \in \Gamma$, $R_L \in \mathcal{R}$ and $LX = S_L R_L$. Furthermore the mapping $L \mapsto R_L$ is a bijection on $\mathcal{R}$.*

Proof. Existence: Since $\mathcal{R}$ is a RRS for Γ, we have $\mathrm{SL}_2(\mathbb{Z}) = \Gamma \cdot \mathcal{R}$ i.e. in particular for the matrix $LX \in \mathrm{SL}_2(\mathbb{Z})$, there are $S \in \Gamma, R \in \mathcal{R}$ such that $LX = SR$.

Uniqueness: Assume there are $S' \in \Gamma, R' \in \mathcal{R}$ with $LX = S'R'$. Then $S'R' = LX = SR$ implies $S^{-1}S'R' = R$ and since $S, S' \in \Gamma$, so is $S^{-1}S'$, so $R \sim R'$ modulo Γ. Since both of them come from a RRS, this implies $R = R'$. With this, $SR = S'R' = S'R$ follows. Canceling out R from both sides yields $S = S'$.

Bijectivity: Since $\mathcal{R}$ is finite, is suffices to show injectivity. Assume therefore that there are L, L' both in $\mathcal{R}$, both mapping to R, so there are $S, S' \in \Gamma$ with

$$\begin{aligned} LX = SR,\ L'X = S'R &\Rightarrow R = S^{-1}LX,\ R = (S')^{-1}L'X \\ &\Rightarrow S^{-1}L = RX^{-1} = (S')^{-1}L' \\ &\Rightarrow S'S^{-1}L = L' \\ &\Rightarrow L \sim L' \quad \text{mod } \Gamma \\ &\Rightarrow L = L' \end{aligned}$$

since both, L and L' come from an RRS. If we set S_L to be the unique S and R_L to be the unique R as constructed above, we are done. $\square$

Proof of Theorem 3.2.1. We have to show the three conditions. (I) is clear as every f_L is holomorphic as a composition of such, hence g is a finite product of holomorphic functions on $\mathbb{H}$, hence holomorphic.

(II): Let $X \in \mathrm{SL}_2(\mathbb{Z})$ arbitrary and fixed. For every $L \in \mathcal{R}$, we obtain S_L, R_L as in the lemma above. With this we have:

$$\begin{aligned}
g(Xz) &= \prod_{L\in\mathcal{R}} f_L(Xz) \\
&= \prod_{L\in\mathcal{R}} \frac{1}{\mu(L:Xz)} f(LTz) \\
&= \prod_{L\in\mathcal{R}} \frac{1}{\mu(L:Xz)} \prod_{L\in\mathcal{R}} f(S_L R_L z) \\
&= \prod_{L\in\mathcal{R}} \frac{1}{\mu(L:Xz)} \prod_{L\in\mathcal{R}} \mu(S_L, R_L z) \frac{\mu(R_L,z)}{\mu(R_L,z)} f(R_L z) \\
&\qquad \text{(as } f \text{ satisfies (II) and } S_L \in \Gamma) \\
&= \prod_{L\in\mathcal{R}} \frac{1}{\mu(L:Xz)} \prod_{L\in\mathcal{R}} \mu(S_L, R_L z)\mu(R_L,z) f_{R_L}(z) \\
&= \prod_{L\in\mathcal{R}} \frac{1}{\mu(L:Xz)} \prod_{L\in\mathcal{R}} \mu(S_L, R_L z)\mu(R_L,z) \underbrace{\prod_{L\in\mathcal{R}} f_{R_L}(z)}_{=\prod f_L \text{ as } L\to R_L \text{ is a bijection on } \mathcal{R}} \\
&= \prod_{L\in\mathcal{R}} \frac{\mu(S_L, R_L z)\mu(R_L,z)}{\mu(L:Xz)} f_L(z)
\end{aligned}$$

Proposition 2.4.2 states that $(XY:z) = (X:Yz)(Y:z)$ for all $X, Y \in \mathrm{SL}_2(\mathbb{Z})$. If we take this equation to the k-th power, we obtain the identity for μ in place of (:), i.e. we have $\mu(XY,z) = [(XY:z)]^k = [(X:Yz)(Y:z)]^k = (X:Yz)^k(Y:z)^k = \mu(X,Yz)\mu(Y,z)$ (note that this step is invalid if $k \in \mathbb{R}$ because of the falsity of the exponentiation rule over $\mathbb{C}$). Therefore the fractional term is

$$\begin{aligned}
\frac{\mu(S_L, R_L z)\mu(R_L,z)}{\mu(L:Xz)} &= \frac{\mu(S_L R_L, z)}{\mu(L,Xz)} \\
&= \frac{\mu(LX,z)}{\mu(L,XZ)} \\
&= \frac{\cancel{\mu(L,Xz)}\mu(X,z)}{\cancel{\mu(L,Xz)}} \\
&= \mu(X,z)
\end{aligned}$$

Hence

$$g(Tz) = \prod_{L\in\mathcal{R}} \frac{\mu(S_L, R_L z)\mu(R_L,z)}{\mu(L:Xz)} f_L(z) = \prod_{L\in\mathcal{R}} \mu(X,z) f_L(z) = \mu(X,z)^{|\mathcal{R}|} g(z)$$

The weight is correct because $\mu(X,z)^{|\mathcal{R}|} = ((X:z)^k)^{[\mathrm{SL}_2(\mathbb{Z}):\Gamma]} = (X:z)^{k[\mathrm{SL}_2(\mathbb{Z}):\Gamma]}$.

(III): This condition is actually the easiest one in this case. Since all f_L are holomorphic at infinity, $f_L(i\infty)$ exists for all $L \in \mathcal{R}$ by Theorem 2.4.7, so $\lim_{z\to i\infty} f_L(z) = c_L$ exists. Then, since $\mathcal{R}$ is finite, we have $\lim_{z\to i\infty} g(z) = \lim_{z\to i\infty} \prod f_L(z) = \prod c_L \in \mathbb{C}$, i.e. because Theorem 2.4.7 works in both directions, $g = g_{Id}$ is holomorphic at infinity. This is all we have to do, because in $\mathrm{SL}_2(\mathbb{Z})$, there is only one cusp namely $\infty = Id.\infty$ and by Theorem 2.4.11 we only have to check (III) at the cusps. □

Although we have proven that g is a modular form, we still need to know its order at infinity because this is what we are interested in.

3.2.3 Theorem. *For the setting as in the last theorem, g as constructed, we have*

$$\omega_g(\infty) = \frac{N_1}{n_1} + ... + \frac{N_u}{n_u}$$

Proof. Let $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = u$, i.e. $\mathcal{R} = L_1, ..., L_u$ and $\omega_{f_L}(\infty) = N_L$, i.e. the Laurent series of the function G_L begins with the term carrying the index N_L. Then, since f is a modular form, all f_L have Fourier expansions of the form $\sum_{n=N_L}^{\infty} a_n(L)e^{2\pi inz/n_L}$. For convenience, we set $n_j = n_{L_j}$, $N_j = N_{L_j}$, $A := \mathrm{lcm}(n_1, ..., n_u)$, $A_j = A/n_j$ for $j \in \{1, ..., u\}$, $r := N_1/n_1 + ... + N_u/n_u = (N_1A_1 + ... + N_uA_u)/A$ and $R := 2\pi ir$.

$$\begin{aligned}
g(z) &= \prod_{L\in\mathcal{R}} f_L(z) \\
&= f_{L_1}(z) \cdot ... \cdot f_{L_u}(z) \\
&= \left(\sum_{n=N_1}^{\infty} a_n(L_1)e^{2\pi inz/n_1}\right) \cdot ... \cdot \left(\sum_{n=N_u}^{\infty} a_n(L_u)e^{2\pi inz/n_u}\right) \\
&= e^{2\pi i\frac{N_1}{n_1}z} ... e^{2\pi i\frac{N_u}{n_u}z} \left(\sum_{n=0}^{\infty} \widetilde{a_n}(L_1)e^{2\pi inz/n_1}\right) \cdot ... \cdot \left(\sum_{n=0}^{\infty} \widetilde{a_n}(L_u)e^{2\pi inz/n_u}\right) \\
&\qquad (\text{where } \widetilde{a_n}(L_j) = a_{n+N_j}(L_j)) \\
&= e^{2\pi irz} \sum_{l=0}^{\infty} \sum_{\substack{(l_1,...,l_u)\in\mathbb{N}^u,\\ l_1+...+l_m=l}} \widetilde{a_{l_1}}(L_1)...\widetilde{a_{l_u}}(L_u)e^{2\pi iz(l_1A_1+...+l_uA_u)/A} \\
&= e^{Rz} \sum_{l=0}^{\infty} c_l e^{2\pi ilz/A} \qquad \text{(see below)}
\end{aligned}$$

Where the c_l are sums over terms of the form $\widetilde{a_{l_1}}(L_1) \cdot ... \cdot \widetilde{a_{l_u}}(L_u)$ with $l_1 + ... + l_u = l$. In particular $c_0 = \widetilde{a_0}(L_1) \cdot ... \cdot \widetilde{a_0}(L_u)$ since $0 + ... + 0$ is the only way to sum up to 0 when only natural summands are allowed. In particular

$$c_0 \neq 0 \tag{3.2}$$

as each $\widetilde{a_0}(L_j) = a_{N_j}(L_j) \neq 0$ as it is the first term in the Laurent/Fourier series that occurs. The validity of the last step in the calculation above is justified by "collecting the terms" (this can indeed be made more precise).

We now consider a function $G : B_1(0) \to \mathbb{C}$ with

$$G(t) := \sum_{l=0}^{\infty} c_l t^l$$

In order to analyze this function further, we define the function $q(z)$ to be $q(z) = e^{2\pi i z/A}$. First, we claim that G converges. We observe that the series $\sum_{l=0}^{\infty} c_l e^{2\pi i l z/A} = \sum_{l=0}^{\infty} c_l q(z)^l$ converges absolutely for all $z \in \overline{\mathbb{H}}$. We can do the steps from the calculation above backwards and thus decompose this sum into a finite product of absolutely converging sums (this is the justification for the Cauchy-product-step). Leaving out the e^{Rz}-term does not change the behavior of convergence. For every $t \in B_1(0)$, we find a $z \in \overline{\mathbb{H}}$ such that $q(z) = t$, hence $|G(t)| \leq \sum_{l=0}^{\infty} |c_l||t^l| = \sum_{l=0}^{\infty} |c_l||q(z)^l| < \infty$. By construction we have

$$e^{-Rz} g(z) = \sum_{l=0}^{\infty} c_l e^{2\pi i l z/A} = G(q(z)) \ \forall z \in \mathbb{H} \tag{3.3}$$

As g satisfies condition (II), $g(z+1) = g(U.z) = (0z+1)^k g(z) = g(z)$. Substituting this into (3.3) gives

$$e^{-R} G(q(z)) = e^{-R} e^{-Rz} g(z) = e^{-R(z+1)} g(z+1) = G(q(z+1)) = G(e^{\frac{2\pi i}{A}} q(z))$$

Since we find a preimage for every $t \in B_1(0)$ under q and the rest of the factors in the last identity do not depend on t or z at all, we obtain for every $t \in B_1(0)$ the identity

$$G(t) = e^R G(t e^{\frac{2\pi i}{A}})$$

If we write G back as its power series we see that

$$\sum_{l=0}^{\infty} c_l t^l = G(t) = e^R G(t e^{\frac{2\pi i}{A}}) = e^R \sum_{l=0}^{\infty} c_l t^l e^{\frac{2\pi i l}{A}}$$

Since power series are unique, for all $l \in \mathbb{N}$ the identity $e^R c_l e^{\frac{2\pi i l}{A}} = c_l$ holds. Rewriting this yields

$$[1-(e^{R+\frac{2\pi i l}{A}})]c_l = 0 \tag{3.4}$$

By (3.2), $c_0 \neq 0$ so we substitute $l = 0$ and cancel c_0 in the above in order to deduce

$$1 = e^{R+\frac{2\pi i 0}{A}} = e^R$$

i.e. $2\pi i r \in 2\pi i\mathbb{Z}$ and consequently $r \in \mathbb{Z}$. More generally speaking, if $c_l \neq 0$ then we can cancel out c_l in (3.4) and see that $e^{R+\frac{2\pi i l}{A}} = 1$ hence $2\pi i(r+l/A) \in 2\pi i\mathbb{Z}$. Since r is an integer, $l/A \in \mathbb{Z}$, so actually, G is a power series in $q(z)^A$, i.e. we now know the Fourier expansion of $g = g_{Id}$. It is:

$$g(z) = e^{rz} \underbrace{G(q(z))}_{\text{series in } q(z)^A} = e^{rz} \sum_{l=0}^{\infty} \widehat{c_l} e^{2\pi i l z},$$

in particular

$$\omega_g(\infty) = r = N_1/n_1 + ... + N_u/n_u$$

. □

We now have a first insight on how g distributes the information of 'order' of along the trace of ∞. In order to see completely through all the structure and also how the information is distributed along the trace of a point $z \in \mathbb{H}$, we need the following:

3.2.4 Lemma. *Given a group G and a subgroup H of finite index and a specific element $s \in G$, there are natural numbers $m, \sigma_1, ..., \sigma_m$, elements $l_1, ..., l_n \in G$ and sets $\mathcal{L}_1, ..., \mathcal{L}_m$ such that*

(i) $\sigma_j = \min\{n \in \mathbb{N} \mid s^n \in l_j^{-1} G l_j\}$

(ii) $\mathcal{L}_j = \{l_j s^0, l_j s^1, ..., l_j s^{\sigma_j - 1}\}$

(iii) $\bigcup_{j=1}^m \mathcal{L}_j$ *is a RRS for G/H*

Proof. See [Ra 77], p. 5, Theorem 1.1.2. □

Notation: $\mathbb{E}_2(\Gamma)^C$ ($\mathbb{E}_3(\Gamma)^C$ respectively) stands for the complement of $\mathbb{E}_2(\Gamma)$, ($\mathbb{E}_3(\Gamma)$ respectively) in the all-or-nothing-principle meaning, i.e.

$$\mathbb{E}_2(\Gamma)^C = \{[\![z]\!] \subset \mathbb{H} \mid z = L.i \text{ for some } L \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \Phi(LVL^{-1}) \notin \widehat{\Gamma}\}$$

and analogously

$$\mathbb{E}_3(\Gamma)^C = \{[\![z]\!] \subset \mathbb{H} \mid z = L.\varrho \text{ for some } L \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \Phi(LPL^{-1}) \notin \widehat{\Gamma}\}.$$

The following is the main part of the proof of the $k/12$ – Formula for subgroups:

3.2.5 Theorem. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ with a finite index, f a modular form for Γ unequal to the zero form having g as an associated modular form for $SL_2(\mathbb{Z})$ as in Theorem 3.2.1, then we have*

$$\frac{1}{c}\omega_g(\infty) = \sum_{\zeta \in \mathrm{cu}(\Gamma)} \frac{\widehat{n_\zeta}}{n_\zeta}\omega_f(\zeta)$$

$$\frac{1}{2c}\omega_g(i) = \frac{1}{2}\left(\sum_{\zeta \in \mathrm{tr}(i)\cap\mathbb{E}_2(\Gamma)} \omega_f(\zeta)\right) + \sum_{\zeta \in \mathrm{tr}(i)\cap\mathbb{E}_2(\Gamma)^C} \omega_f(\zeta)$$

$$\frac{1}{3c}\omega_g(\varrho) = \frac{1}{3}\left(\sum_{\zeta \in \mathrm{tr}(\varrho)\cap\mathbb{E}_3(\Gamma)} \omega_f(\zeta)\right) + \sum_{\zeta \in \mathrm{tr}(\varrho)\cap\mathbb{E}_3(\Gamma)^C} \omega_f(\zeta)$$

and for $z \in \mathbb{F}_I$ with $z \not\approx i, \varrho$ we have

$$\frac{1}{c}\omega_g(z) = \sum_{\zeta \in \mathrm{tr}(z)} \omega_f(\zeta)$$

where $c = 2$ if $-Id \in \Gamma$ and $c = 1$ otherwise.

Proof. We know that the stabilizer $\widehat{\mathrm{SL}_2(\mathbb{Z})}_z$ is cyclic (thm 2.2.10). We choose S to be its generator and apply the above lemma with $G = \widehat{\mathrm{SL}_2(\mathbb{Z})}, H = \widehat{\Gamma}, s = S$ in order to obtain an RRS

$$\mathcal{R} = \{\underbrace{L_1, L_1S, ..., L_1S^{\sigma_1-1}}_{=\mathcal{L}_1}, \underbrace{L_2, L_2S, ..., L_2S^{\sigma_2-1}}_{=\mathcal{L}_2}, ..., L_mS^{\sigma_m-1}\}.$$

Note that this is an RRS of $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$ and not of $\mathrm{SL}_2(\mathbb{Z})/\Gamma$! We compute the trace of z to be

$$\begin{aligned}\operatorname{tr}(z) &= \mathrm{SL}_2(\mathbb{Z}).z/\approx \\ &= \widehat{\mathrm{SL}_2(\mathbb{Z})}.z/\approx \\ &= \widehat{\Gamma}.\mathcal{R}.z/\approx \\ &= \{\widehat{\Gamma}.L_1.z, \widehat{\Gamma}.L_1.\underbrace{Sz}_{=z}, ..., \widehat{\Gamma}.L_1.\underbrace{S^{\sigma_1-1}.z}_{=z}, ...\}/\approx \\ &= \{\widehat{\Gamma}.L_1.z, \widehat{\Gamma}.L_2.z, ..., \widehat{\Gamma}.L_m.z\}/\approx \\ &= \{\left[\!\left[\widehat{\Gamma}.L_1.z\right]\!\right], ..., \left[\!\left[\widehat{\Gamma}.L_m.z\right]\!\right]\} \\ &= \{[\![L_1.z]\!], ..., [\![L_m.z]\!]\}\end{aligned}$$

And the points $L_i.z$ and $L_j.z$ are incongruent modulo Γ for $i \neq j$, because

$$\begin{aligned}L_i.z \approx L_j.z \mod \Gamma &\Rightarrow \exists\, \overline{T} \in \Gamma \text{ s.t. } L_i z = \overline{T} L_j z \\ &\Rightarrow L_i^{-1}\overline{T}L_j z = z \\ &\Rightarrow L_i^{-1}TL_j \in \widehat{\mathrm{SL}_2(\mathbb{Z})}_z \\ &\qquad \text{for } T = \Phi(\overline{T}) \\ &\Rightarrow L_i^{-1}TL_j \in \widehat{\mathrm{SL}_2(\mathbb{Z})}_z = \langle S \rangle \\ &\Rightarrow L_i^{-1}TL_j = S^n \\ &\qquad \text{for some } n = d\sigma_i + r \in \mathbb{N} \text{ with } d, r \in \mathbb{N}, 0 \leq r < \sigma_i \\ &\Rightarrow TL_j = L_i S^{d\sigma_j} S^r = \underbrace{L_i S^{d\sigma_j} L_i^{-1}}_{=(L_i S^{\sigma_j} L_i^{-1})^d \in \widehat{\Gamma} \text{ by 3.2.4 (i)}} L_i S^r \\ &\Rightarrow L_i S^r = (L_i S^{\sigma_j} L_i^{-1})^{-d} T L_j \in \widehat{\Gamma} \cdot L_j \\ &\Rightarrow (\text{since } r < \sigma_i)\ L_j \sim L \text{ for some } L \in \mathcal{L}_i\end{aligned}$$

This is a contradiction as the union of the $\mathcal{L}_i$ forms an RRS so in particular, all the matrices from the sets $\mathcal{L}_i, \mathcal{L}_j$ for $i \neq j$ are incongruent modulo $\widehat{\Gamma}$.

We have derived the equation

$$\sum_{\zeta \in \operatorname{tr}(z)} \omega_f(\zeta) = \sum_{\zeta \in \{[\![L_1.z]\!], ..., [\![L_m.z]\!]\}} \omega_f(\zeta) = \sum_{j=1}^{m} \omega_f(L_j.z) \tag{3.5}$$

and if we define $n_j := n_{L_j}, \widehat{n_j} := \widehat{n_{L_j}}$ then we have moreover derived that

$$\sum_{\zeta \in \operatorname{tr}(\infty)} \frac{\widehat{n_\zeta}}{n_\zeta} \omega_f(\zeta) = \sum_{\zeta \in \{[\![L_1.z]\!], ..., [\![L_m.z]\!]\}} \frac{\widehat{n_j}}{n_j} \omega_f(\zeta) = \sum_{j=1}^{m} \frac{\widehat{n_j}}{n_j} \omega_f(L_j.z) \tag{3.6}$$

We now make a case distinction on whether $z = \infty, z = i, z = \varrho, z \in \mathbb{H} \setminus \{\infty, i, \varrho\}$. In each case we must clarify the role of σ_i first:

<u>1st case: $z = \infty$</u> The generator of $\widehat{\mathrm{SL}_2(\mathbb{Z})}_\infty$ is $S = \Phi(U) = \widehat{U}$ in this case. Setting $S_j = L_j(\widehat{U}^n)L_j^{-1}$ to be the generator of $\widehat{\mathrm{SL}_2(\mathbb{Z})}_{L_j.\infty}$ and noting that $\widehat{U}^n \in L_j^{-1}\widehat{\Gamma}L_j \iff L_j(\widehat{U}^n)L_j^{-1} \in \widehat{\Gamma}$ leads to

$$\begin{aligned}\sigma_j &= \min\{n \in \mathbb{N} \mid \widehat{U}^n \in L_j^{-1}\widehat{\Gamma}L_j\} \\ &= \min\{n \in \mathbb{N} \mid L_j(\widehat{U}^n)L_j^{-1} \in \widehat{\Gamma}\} \\ &= \min\{n \in \mathbb{N} \mid S_j^n \in \widehat{\Gamma}\} \\ &= \widehat{n_{L_j}} = \widehat{n_j} \end{aligned} \tag{3.7}$$

Also note that

$$n_{L_j} = n_{L_jS} = ... = n_{L_jS^{\sigma_j-1}}, \tag{3.8}$$

because $L_j.\infty = L_j.S.\infty = ... = L_j.S^{\sigma_j-1}.\infty$ – solely because $S.\infty = \infty$ – and n_x does only depend on the point x. All in all we obtain:

$$\begin{aligned}\sum_{\zeta\in\mathrm{tr}(\infty)} \frac{\widehat{n_\zeta}}{n_\zeta}\omega_f(\zeta) &= \sum_{j=1}^{m} \frac{\widehat{n_j}}{n_j}\omega_f(L_j.z) && \text{(from (3.6))} \\ &= \sum_{j=1}^{m} \frac{\widehat{n_j}}{n_j}\omega_f(L_j.z) \\ &= \sum_{j=1}^{m}\sum_{d=0}^{\sigma_j-1} \frac{1}{n_j}\omega_f(L_j.\underbrace{z}_{=S^d.z}) && (\widehat{n_j} = \sigma_j \text{ by (3.7)}) \\ &= \sum_{j=1}^{m}\sum_{d=0}^{\sigma_j-1} \frac{1}{n_{L_jS^d}}\omega_f(L_j.S^d.z) \\ &\quad \text{(as } n_j = n_{L_j} = n_{L_jS^d} \text{ for all } d \text{ by (3.8))} \\ &= \sum_{L\in\mathcal{R}} \frac{\omega_f(Lz)}{n_L} \\ &= \sum_{L\in\mathcal{R}} \frac{N_L}{n_L} && (3.9)\end{aligned}$$

The last line in (3.9) does not depend on the concrete RRS: given $\mathcal{R}$ and $\widetilde{\mathcal{R}}$, the mapping $L \in \mathcal{R} \mapsto$ [the $\widetilde{L}$ in $\widetilde{\mathcal{R}}$ with $L = T\widetilde{L}$ for some $T \in \Gamma$] is a bijection and N_L, n_L do only depend on the orbit in the sense that $n_{T\widetilde{L}} = n_{\widetilde{L}}, N_{T\widetilde{L}} = N_{\widetilde{L}}$, see 3.0.2.

Consider the fixed RRS $\mathcal{R}' = \{L'_1, ..., L'_u\}$ used to construct g from f in Theorem 3.2.1. We can consider the homogenized version

$$\widehat{\mathcal{R}} = \Phi(\mathcal{R}') = \{\Phi(L'_1), ..., \Phi(L'_u)\}$$

Note that this is *not necessarily* a RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$. Depending on whether or not $-Id \in \Gamma$, there are two possibilities: If $-Id \in \Gamma$, then $\widehat{\mathcal{R}}$ actually is an RRS by 2.1.8(c). We also have $|\Phi(\mathcal{R}')| = |\mathcal{R}'|$ and

$$\Phi(L'_1).\infty = L'_1.\infty, ..., \Phi(L'_u).\infty = L'_u.\infty$$

and therefore

$$n_{\Phi(L'_j)} = n_{\Phi(L'_j).\infty} = n_{L'_j.\infty}, \quad N_{\Phi(L'_j)} = N_{\Phi(L'_j).\infty} = N_{L'_j.\infty},$$

see 2.3.5 and 2.4.10. Therefore, in the last sum of (3.9), we can switch from $\mathcal{R}$ to $\Phi(\mathcal{R}')$ (as both are RRS for the homogeneous versions) and then from $\Phi(\mathcal{R}')$ to $\mathcal{R}'$ (as they coincide in terms of amount of members and movements) and obtain

$$\sum_{\zeta \in \mathrm{tr}(\infty)} \frac{\widehat{n_\zeta}}{n_\zeta} \omega_f(\zeta) = \sum_{L \in \mathcal{R}} \frac{N_L}{n_L} = \sum_{L' \in \mathcal{R}'} \frac{N_{L'}}{n_{L'}} = \frac{N_{L'_1}}{n_{L'_1}} + ... + \frac{N_{L'_u}}{n_{L'_u}} = \omega_g(\infty)$$

As $\omega_g(\infty) = \frac{N_{L'_1}}{n_{L'_1}} + ... + \frac{N_{L'_u}}{n_{L'_u}}$ by Theorem 3.2.3.

If $-Id \notin \Gamma$, then things look different: by Lemma 2.1.8, u is even and precisely half of the L collide in the sense that we find $G_{\frac{u}{2}+1}, ..., G_u$ all in Γ such that

$$L'_{\frac{u}{2}+1} = G_1(-L'_1), ..., L'_u = G_{\frac{u}{2}}(-L'_{\frac{u}{2}})$$

and $\widehat{\mathcal{R}} := \{\Phi(L_1), ..., \Phi(L_{\frac{u}{2}})\}$ forms an RRS for $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$. When switching from $\mathcal{R}$ to $\widehat{\mathcal{R}}$ we see that half of the summands disappear. However, since

$$n_{L_{(\frac{u}{2}+j)}} = n_{G_{(\frac{u}{2}+j)}(-L'_j)} \overset{3.0.2}{=} n_{-L'_j} \overset{2.3.5}{=} n_{-L'_j.z} \overset{2.3.5}{=} n_{\Phi(L'_j)} \tag{3.10}$$

and

$$N_{L'_{(\frac{u}{2}+j)}} = N_{G_{(\frac{u}{2}+j)}(-L'_j)} \overset{3.0.2}{=} N_{-L'_j} \overset{2.4.10}{=} N_{-L'_j.z} \overset{2.4.10}{=} N_{\Phi(L'_j)} \tag{3.11}$$

for all $j = 1, ..., u/2$, we can put them into the sum again:

$$\sum_{\zeta \in \mathrm{tr}(\infty)} \frac{\widehat{n_\zeta}}{n_\zeta} \omega_f(\zeta) = \sum_{L \in \mathcal{R}} \frac{N_L}{n_L} = \sum_{L \in \widehat{\mathcal{R}}} \frac{N_L}{n_L}$$

$$= \frac{1}{2} \cdot 2 \cdot \left[\frac{N_{\Phi(L'_1)}}{n_{\Phi(L'_1)}} + ... + \frac{N_{\Phi(L'_{\frac{u}{2}})}}{n_{\Phi(L'_{\frac{u}{2}})}} \right]$$

$$= \frac{1}{2} \cdot \left[\frac{N_{\Phi(L'_1)}}{n_{\Phi(L'_1)}} + ... + \frac{N_{\Phi(L'_{\frac{u}{2}})}}{n_{\Phi(L'_{\frac{u}{2}})}} + \frac{N_{\Phi(L'_1)}}{n_{\Phi(L'_1)}} + ... + \frac{N_{\Phi(L'_{\frac{u}{2}})}}{n_{\Phi(L'_{\frac{u}{2}})}} \right]$$

$$= \frac{1}{2} \cdot \left[\frac{N_{L'_1}}{n_{L'_1}} + ... + \frac{N_{L'_{\frac{u}{2}}}}{n_{L'_{\frac{u}{2}}}} + \frac{N_{L'_{(\frac{u}{2}+1)}}}{n_{L'_{(\frac{u}{2}+1)}}} + ... + \frac{N_{L'_u}}{n_{L'_u}} \right]$$

(by (3.10), (3.11))

$$= \frac{1}{2} \omega_g(\infty)$$

2nd case: $z = i$ Here, $S = \pm V$ and σ_j takes the role of the "opposite" of the size of the stabilizer $\widehat{\Gamma}_{L_j.i}$ meaning that

$$\sigma_j = 1 \iff \Phi(L_j V L_j^{-1}) \in \widehat{\Gamma} \iff [\![L_j.z]\!] \in \mathbb{E}_2(\Gamma) \iff |\widehat{\Gamma}_{L_j.i}| = 2$$

and

$$\sigma_j = 2 \iff \Phi(L_j V L_j^{-1}) \notin \widehat{\Gamma} \iff [\![L_j.z]\!] \notin \mathbb{E}_2(\Gamma) \iff |\widehat{\Gamma}_{L_j.i}| = 1$$

because of the all-or-nothing-principle of the stabilizers (see section 2.2). Moreover, no other value can be taken by σ_j apart from $1, 2$ as $\Phi(L_j V L_j^{-1})$ is of order two as V is. After we resorted the σ_j, we can assume that $\sigma_1 = ... = \sigma_r = 1$ and $\sigma_{r+1} = ... = \sigma_m = 2$ (i.e. iff. $L_1.i, ..., L_r.i$ are all contained in $\mathbb{E}_2(\Gamma)$ and $L_{r+1}.i, ..., L_m.i$ are not), then

$$\mathcal{R} = \{L_1, ..., L_r, L_{r+1}, L_{r+1}V, ..., L_m, L_m V\}. \tag{3.12}$$

Again, let $\mathcal{R}'$ be the RRS that was used to construct g from f and let

$-Id \in \Gamma$. Computing the order of g at $z = i$ we see that

$$\begin{aligned}
\frac{1}{2}\omega_g(i) &= \frac{1}{2}\omega_{(\prod_{L\in\mathcal{R}'} f_L)}(i) \\
&= \frac{1}{2}\sum_{L\in\mathcal{R}'} \omega_{f_L}(i) \\
&= \frac{1}{2}\sum_{L\in\mathcal{R}'} \underbrace{\omega_{1/(L:z)^k}(i)}_{=0} + \omega_{(f\circ L.)}(i) \\
&= \frac{1}{2}\sum_{L\in\mathcal{R}'} \omega_f(L.i) \\
&\quad \text{(see Lemma 3.0.1 and cp. Lemma 3.0.2(i))} \\
&= \frac{1}{2}\sum_{L\in\mathcal{R}} \omega_f(L.i) \\
&\quad \text{(term is independent of RRS, cp. case } z=\infty\text{)} \\
&= \frac{1}{2}\big[\omega_f(L_1.i) + ... + \omega_f(L_r.i) + \\
&\qquad \omega_f(L_{r+1}.i) + \omega_f(L_{r+1}\underbrace{V.i}_{=i}) + ... + \omega_f(L_m.i) + \omega_f(L_m\underbrace{V.i}_{=i})\big] \\
&= \frac{1}{2}\sum_{\substack{j\in\{1,\dots,m\},\\ L_j.i\in\mathbb{E}_2(\Gamma)}} \omega_f(L_j.i) + \frac{1\!\!/}{2\!\!/}\cdot 2\!\!/\cdot \sum_{\substack{j\in\{1,\dots,m\},\\ L_j.i\notin\mathbb{E}_2(\Gamma)}} \omega_f(L_j.i) \\
&= \frac{1}{2}\sum_{\substack{\zeta\in\mathrm{tr}(i),\\ \zeta\in\mathbb{E}_2(\Gamma)}} \omega_f(\zeta) + \sum_{\substack{\zeta\in\mathrm{tr}(i),\\ \zeta\notin\mathbb{E}_2(\Gamma)}} \omega_f(\zeta)
\end{aligned} \tag{3.13}$$

If $-Id \notin \Gamma$, analogously to the case $z = \infty$, we obtain

$$\begin{aligned}
\frac{1}{2c}\omega_g(i) &= \frac{1}{2}\frac{1}{2}\sum_{L\in\mathcal{R}'} \omega_f(L.i) \\
&= \frac{1}{2}\sum_{L\in\mathcal{R}} \omega_f(L.i) \\
&= \frac{1}{2}\sum_{\substack{\zeta\in\mathrm{tr}(i),\\ \zeta\in\mathbb{E}_2(\Gamma)}} \omega_f(\zeta) + \sum_{\substack{\zeta\in\mathrm{tr}(i),\\ \zeta\notin\mathbb{E}_2(\Gamma)}} \omega_f(\zeta)
\end{aligned}$$

3rd case: $z = \varrho$ Completely analogous to the case $z = i$ with the factors 1/3 instead of 1/2.

4th case: $z \not\approx i, \varrho$: Here, $S = \Phi(ID)$ as $\widehat{\mathrm{SL}_2(\mathbb{Z})}_z = \{\Phi(Id)\}$ (see thm. 2.2.10) so $\sigma_j = 1$ for all j and thus, as in (3.13)

$$\begin{aligned}\omega_g(z) &= \sum_{L\in\mathcal{R}'} \omega_f(L.z)\\ &= \frac{1}{c}\sum_{L\in\mathcal{R}} \omega_f(L.z)\\ &= \frac{1}{c}\sum_{\zeta\in\mathrm{tr}(z)} \omega_f(\zeta)\end{aligned}$$

Where $c = 2$ if $-Id \notin \Gamma$ and 1 otherwise. □

For convenience we define the set

$$\mathcal{J} := \{\zeta \in \mathbb{H}/\approx \ \mid \exists z \in \mathbb{F}_I \ : \ \zeta \in \mathrm{tr}(z), \zeta \notin \mathbb{E}_2(\Gamma), \zeta \notin \mathbb{E}_3(\Gamma)\}$$

We have therefore shown the $k/12-$Formula for modular forms of integer weight for arbitrary subgroups of finite index (compare with [Sko 92], note that in the notation used there, $\mathrm{ord}_{L.\infty}(f) = \omega_f(L.\infty)/n_L$):

3.2.6 Theorem ($k/12-$Formula). *Let Γ be a subgroup having finite index in $SL_2(\mathbb{Z})$ and f be a modular form of integer weight k for Γ unequal to the zero form, then the following identity holds:*

$$\sum_{\zeta\in\mathrm{cu}(\Gamma)} \frac{\widehat{n_\zeta}}{n_\zeta}\omega_f(\zeta) + \frac{1}{2}\sum_{\zeta\in\mathbb{E}_2(\Gamma)} \omega_f(\zeta) + \frac{1}{3}\sum_{\zeta\in\mathbb{E}_3(\Gamma)} \omega_f(\zeta) + \sum_{\zeta\in\mathcal{J}} \omega_f(\zeta) = \frac{k[\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma}]}{12}$$

Proof. Construct g according to Theorem 3.2.1, then apply the usual $k/12-$Formula for modular forms for $\mathrm{SL}_2(\mathbb{Z})$ and then apply Theorem 3.2.5 to the four terms in the total order of g to obtain complete description of the orders of f:

$$\sum_{\zeta\in\mathrm{cu}(\Gamma)} \frac{\widehat{n_\zeta}}{n_\zeta}\omega_f(\zeta) + \frac{1}{2}\sum_{\zeta\in\mathbb{E}_2(\Gamma)} \omega_f(\zeta) + \frac{1}{3}\sum_{\zeta\in\mathbb{E}_2(\Gamma)} \omega_f(\zeta) + \sum_{\zeta\in\mathcal{J}} \omega_f(\zeta) = \frac{\frac{k}{c}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma]}{12}$$

And this is the identity claimed as

$$\frac{1}{c}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = \begin{cases}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = [\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}] & \text{if } -Id \in \Gamma\\ \frac{1}{2}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma] = \frac{\not 2}{\not 2} \cdot [\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}] & \text{otherwise}\end{cases}$$

by Lemma 2.1.8. □

Remark the right hand side depends on the index of the *homogenized* versions of the groups and not the index itself. This is due to the fact that we choose $\mathcal{R}$ to be an RRS of $\widehat{\mathrm{SL}_2(\mathbb{Z})}/\widehat{\Gamma}$ and *not* of $\mathrm{SL}_2(\mathbb{Z})/\Gamma$.

3.2.7 Remark. *Whenever we speak of the $k/12-$ Formula, we mean the version for modular forms of integer weight for arbitrary subgroups of finite index. The result can be proven to be correct even for weaker definitions of the term "modular form" (see [Ra 77]). The term on the left is called the total order of f, for short:* $\operatorname{tot}(f)$.

3.3 A first consequence of the $k/12-$ Formula

As mentioned in the introduction, we need three ingredients to show that the modular forms of integer weight form a polynomial ring. Point ② was an upper bound of the dimension of the space of these forms. This is what we will obtain from the $k/12-$ Formula now. We will heavily rely on the assumption that all modular forms are holomorphic, i.e. $\omega_f(z) \geq 0 \; \forall z \in \overline{\mathbb{H}}$.

3.3.1 Definition. *Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$. We define the vector space $M_k(\Gamma)$ to be*

$$M_k(\Gamma) := \{f \mid f \quad \text{is a modular form for } \Gamma \text{ of weight } k\}.$$

That $M_k(\Gamma)$ actually is a vector space can be shown by some direct computations. We can give an upper bound on the dimension of $M_k(\Gamma)$:

3.3.2 Theorem (Dimension formula)**.** *Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$, then for any $k \in \mathbb{Z}$ we have*

$$\dim(M_k(\Gamma)) \leq \begin{cases} 0 & \text{if } k < 0 \\ 1 + \left\lfloor \frac{k[\widehat{SL_2(\mathbb{Z})}:\widehat{\Gamma}]}{12} \right\rfloor & \text{if } k \geq 0 \end{cases}$$

Proof. The first case is clear as the right hand side of the $k/12-$ Formula is always nonnegative (but the right hand side – which equals the left hand side – would be if there was any modular form with negative weight). Set

$$n := 1 + \left\lfloor \frac{k[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}]}{12} \right\rfloor .$$

Let $x_1, ..., x_n$ be points in $\mathbb{H}$ that are neither congruent to i nor to ϱ modulo Γ. Finding such points is easy: As $\mathbb{F}_I$ is a proper fundamental domain, we have an uncountable stock of points that are incongruent to i and ϱ modulo $\mathrm{SL}_2(\mathbb{Z})$ (we may simply select n points from $\mathbb{F}_I$ *unequal* to i and ϱ). Since a congruency modulo a subgroup is in particular a congruency modulo $\mathrm{SL}_2(\mathbb{Z})$, they are also incongruent to i, ϱ modulo any subgroup Γ. The homomorphism of vector spaces

$$\Psi : M_k(\Gamma) \to \mathbb{C}^n, \Psi(f) = (f(x_1), ..., f(x_n))$$

is injective. Indeed, if $f(x_1) = ... = f(x_n) = 0$ but f is not the zero form then $\omega_f(x_1) \geq 1, ..., \omega_f(x_n) \geq 1$ and as they are all incongruent to i and ϱ, these terms are summed up in the last sum, so

$$\begin{aligned}
\mathrm{tot}(f) &= \underbrace{\sum_{\zeta\in\mathrm{cu}(\Gamma)} \frac{\widehat{n_\zeta}}{n_\zeta}\omega_f(\zeta)}_{\geq 0} + \underbrace{\frac{1}{2}\sum_{\zeta\in\mathbb{E}_2(\Gamma)} \omega_f(\zeta)}_{\geq 0} + \underbrace{\frac{1}{3}\sum_{\zeta\in\mathbb{E}_2(\Gamma)} \omega_f(\zeta)}_{\geq 0} + \sum_{z\in\mathcal{J}} \omega_f(\zeta) \\
&\geq \sum_{z\in\mathcal{J}} \omega_f(\zeta) \\
&\geq \underbrace{\omega_f(x_1)}_{\geq 1} + ... + \underbrace{\omega_f(x_n)}_{\geq 1} \\
&\geq n \\
&> \frac{k[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}]}{12} \\
&= \mathrm{tot}(f) \\
&\quad \text{(by the } k/12 - \text{Formula)}
\end{aligned}$$

a contradiction. □

3.3.3 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ having finite index. A modular form f for Γ is called a* cusp form *if it "vanishes at all cusps", i.e. if for every cusp $L.\infty$, $\omega_f(L.\infty) > 0$. The vector space of all cusp forms of weight k will be referred to as $S_k(\Gamma)$.*

3.3.4 Theorem (Dimension formula for cusp forms)**.** *Let Γ be a subgroup of finite index in $SL_2(\mathbb{Z})$, $c = |\operatorname{cu}(\Gamma)|$ its total amount of cusp orbits and r be the amount of regular cusp orbits. Then for any $k \in \mathbb{Z}$ we have*

$$\dim(S_k(\Gamma)) \leq \begin{cases} 0 & \textit{if } 0 \leq k < \frac{r+c}{2}\frac{12}{[\widehat{SL_2(\mathbb{Z})}:\widehat{\Gamma}]} \\ 1 + \left\lfloor \frac{k[\widehat{SL_2(\mathbb{Z})}:\widehat{\Gamma}]}{12} - \frac{r+c}{2} \right\rfloor & \textit{else} \end{cases}$$

Proof. If $k < 0$ then there is no modular form at all and in particular no cusp form apart from the zero form by the usual dimension formula. Let $k \in \mathbb{N}$ and set

$$n := \begin{cases} 0 & \text{if } 0 \leq k < \frac{r+c}{2} \frac{12}{[\widehat{\mathrm{SL}_2(\mathbb{Z})}:\widehat{\Gamma}]} \\ 1 + \left\lfloor \frac{k[\widehat{\mathrm{SL}_2(\mathbb{Z})}:\widehat{\Gamma}]}{12} - \frac{r+c}{2} \right\rfloor & \text{else} \end{cases}$$

As in Theorem 3.3.2 we find points $x_1, ..., x_n$ in $\mathbb{H}$ (in the first case we do not need any point at all) that are neither congruent to i nor to ϱ modulo Γ. Again we consider the homomorphism of vector spaces

$$\Psi : M_k(\Gamma) \to \mathbb{C}^n, \Psi(f) = (f(x_1), ..., f(x_n))$$

and show that it is injective: Assume $f(x_1) = ... = f(x_n) = 0$ but f not being the zero form, then the $k/12-$Formula is applicable. Note that since ζ_j is regular for $j \leq r$, we have $\widehat{n_{\zeta_j}} = n_{\zeta_j}$ by definition of regularity in this case whence for $j > r$, by 2.3.8, we have $\widehat{n_{\zeta_j}}/n_{\zeta_j} = 1/2$. This leads to

$$\begin{aligned} \mathrm{tot}(f) &= \sum_{\zeta \in \mathrm{cu}(\Gamma)} \frac{\widehat{n_\zeta}}{n_\zeta} \omega_f(\zeta) + \underbrace{\frac{1}{2} \sum_{\zeta \in \mathbb{E}_2(\Gamma)} \omega_f(\zeta)}_{\geq 0} + \underbrace{\frac{1}{3} \sum_{\zeta \in \mathbb{E}_2(\Gamma)} \omega_f(\zeta)}_{\geq 0} + \underbrace{\sum_{\zeta \in \mathcal{J}} \omega_f(\zeta)}_{\geq 0} \\ &\geq \sum_{i=1}^{r} \omega_f(\zeta_i) + \sum_{i=r+1}^{c} \frac{1}{2} \omega_f(\zeta_i) + n \\ &\geq r + \frac{c - (r+1) + 1}{2} + n \\ &= \frac{r+c}{2} + n \\ &> \frac{k[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma}]}{12} \\ &= \mathrm{tot}(f) \end{aligned}$$

a contradiction. $\square$

4 The polynomial structure of the rings of modular forms

In this section we finally want to show that the vector space

$$M_*(\Gamma) := \sum_{k \in \mathbb{Z}} M_k(\Gamma)$$

is equal to the set $\mathbb{C}[f_0, g_0]$ where f_0, g_0 are modular forms for the subgroup Γ for $\Gamma \in \{\Gamma_0(2), \Gamma_1(3)\}$. In other words: the ring spanned by all modular forms of *integer* weight of these subgroups forms a polynomial ring in two fixed modular forms. Note that $M_*(\Gamma)$ is *not* the set of all modular forms of integer weight. This set is contained in $M_*(\Gamma)$, but indeed, $M_*(\Gamma)$ also contains sums of modular forms of different weights which are not again modular forms because every summand transforms in its own weight.

A direct computation shows that $M_*(\Gamma)$ can be enriched by the usual multiplication: if f, g are modular forms of weight k, l then $f \cdot g$ transforms like

$$(f \cdot g)(Tz) = f(Tz)g(Tz) = (T:z)^k (T:z)^l (f \cdot g)(z) = (T:z)^{k+l}(f \cdot g)(z)$$

and since we have characterized the holomorphicity at the cusps via limits in Theorem 2.4.7, we see that $f \cdot g$ is holomorphic at the cusps solely because for every cusp $L.\infty$,

$$\begin{aligned}
\lim_{z\to i\infty} (f \cdot g)_L(z) &= \lim_{z\to i\infty} \frac{1}{(L:z)^{k+l}} f(Lz) \cdot g(Lz) \\
&= \lim_{z\to i\infty} \frac{1}{(L:z)^{k}} f(Lz) \cdot \lim_{z\to i\infty} \frac{1}{(L:z)^{l}} g(Lz) \\
&= \lim_{z\to i\infty} f_L(z) \cdot \lim_{z\to i\infty} g_L(z) \\
&= f_L(i\infty) g_L(i\infty)
\end{aligned}$$

exists. We have therefore already shown that $M_*(\Gamma)$ is not only a vector space but also a ring with multiplication defined component wise, i.e.

$$(\lambda_1 f_1 + \ldots + \lambda_n f_n) * (\mu_1 g_1 + \ldots + \mu_m g_m) := \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} \lambda_i \mu_j \underbrace{(f_i \cdot g_j)}_{\in M_*(\Gamma)}$$

and since everything behaves as usual we will just write the symbol "$\cdot$" in place of "$*$". From the point of vector spaces we now first decompose M_* into smaller spaces:

4.0.1 Theorem. *Assume that in Γ there is a matrix $T = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ such that $c \neq 0$ then $M_*(\Gamma)$ is a graded ring, i.e.*

$$M_*(\Gamma) = \bigoplus_{k\in\mathbb{N}} M_k(\Gamma)$$

Proof. Let $f_1, ..., f_n$ be modular forms all unequal to the zero form coming from pairwise different spaces $M_{k_i}(\Gamma)$, i.e. f_i is a modular form of weight k_i and $k_i \neq k_j$ for $i \neq j$. Formally we prove the claim by induction on n but we will present it in a step by step way as this is more readable. After resorting the f_i we have $k_1 > k_2 > ... > k_n$. Assume we have a nontrivial combination

$$\sum_{i=1}^{n} \lambda_i f_i(z) = 0 \ \forall \ z \in \mathbb{H} \tag{i}$$

As this is a functional equation that holds for all $z \in \mathbb{H}$ we also may substitute Tz for z. Since all the f_i satisfy condition (II), this yields the functional equation

$$0 = \sum_{i=1}^{n} \lambda_i f_i(z) = \sum_{i=1}^{n} \lambda_i f_i(Tz) = \sum_{i=1}^{n} \lambda_i (cz+d)^{k_i} f_i(z) \ \forall \ z \in \mathbb{H} \tag{ii}$$

from this we obtain

$$\begin{aligned}
0 &= (cz+d)^{k_1}(i) - (ii) \\
&= \cancel{\lambda_1 (cz+d)^{k_1} f_1(z)} + \sum_{i=2}^{n} \lambda_i (cz+d)^{k_1} f_i(z) \\
&\quad - \cancel{\lambda_1 (cz+d)^{k_1} f_1(z)} - \sum_{i=2}^{n} \lambda_i (cz+d)^{k_i} f_i(z) \\
&= \sum_{i=2}^{n} \lambda_i [(cz+d)^{k_i + (k_1 - k_i)} - (cz+d)^{k_i}] f_i(z) \\
&= \sum_{i=2}^{n} \lambda_i (cz+d)^{k_i} [(cz+d)^{k_1 - k_i} - 1] f_i(z) \qquad \text{(iii)}
\end{aligned}$$

Continuing by substituting Tz for z i.e. $f_i(Tz) = (cz+d)^{k_i} f_i(z)$ in (iii) yields

$$0 = \sum_{i=2}^{n} \lambda_i (cz+d)^{2k_i} [(cz+d)^{k_1 - k_i} - 1] f_i(z) \tag{iv}$$

Computing $0 = 0 - 0 = (cz+d)^{k}_{2} \cdot (iii) - (iv)$, we obtain analogously to the computation of (iii):

$$0 = \sum_{i=3}^{n} \lambda_i \big((cz+d)^{k_i + k_2} - (cz+d)^{2k_i} \big) [(cz+d)^{k_1 - k_i} - 1] f_i(z) \tag{v}$$

Observe how the sum shrinks step by step. Continuing in this way we arrive at the n-th summand with an equation of the form

$$0 = \lambda_n p(z) f_n(z) \tag{2n}$$

where $p(z)$ is some polynomial in z having some power of c as exponent of the leading term. Here we use that $c \neq 0$ so p is not constant hence it must have a finite number of roots in the field $\mathbb{C}$, say $z_1, ..., z_{\deg(p)}$. If we select any $z_0 \in \mathbb{H}$ unequal to those finitely many roots of p and also unequal to the zeros of f_n (which is possible as the zeros do not accumulate), we obtain

$$0 = \lambda_n \underbrace{p(z_0)}_{\neq 0} \underbrace{f_n(z_0)}_{\neq 0} \tag{2n}$$

and finally that $\lambda_n = 0$. We substitute this into equation nr. $(2n-2)$:

$$0 = \lambda_{n-1}\widetilde{p}(z) f_{n-1}(z) + \lambda_n p(z) f_n(z) = \lambda_{n-1}\widetilde{p}(z) f_{n-1}(z) \tag{2n-2}$$

and obtain in the same manner as above that $\lambda_{n-1} = 0$. Continuing in this way we have shown that all λ_i are equal to zero hence that the sum is direct. $\square$

The sense of this step is to reduce the structural analysis of $M_*(\Gamma)$ to the analysis of the $M_k(\Gamma)$ spaces. In order to examine these further, we will create a certain number of linearly independent modular forms and then use the dimension formula to show that we have indeed found a basis. In order to create a lot of different forms from two basic ones we will now derive a sufficient condition. Before doing that we will need a small definition:

4.0.2 Definition. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$, f a modular form for Γ and $z = L.\infty$ a cusp, then we say that $f(z) = f(L.\infty) = c$ for some $c \in \mathbb{C}$ iff.* $\lim_{z\to i\infty} f_L(z) = c$. *Note that this does* NOT *mean that also* $\lim_{z\to L.\infty\in\mathbb{P}} f(z) = c$ *we just use this as an abbreviation.*

4.0.3 Theorem. *Let Γ be a subgroup of $SL_2(\mathbb{Z})$ and f, g modular forms of weights $k_1, k_2 \in \mathbb{N} \setminus \{0\}$ and let $z_1, z_2 \in \overline{\mathbb{H}}$ such that f and g behave "differently" at these two points, i.e.*

$$f(z_1) \neq 0,\ g(z_1) = 0,\ f(z_2) = 0,\ g(z_2) \neq 0$$

Then these two modular forms are algebraically independent in the sense that for every polynomial $P(x, y) \in \mathbb{C}[x, y]$, $P(f, g) \equiv$ zero *on $\mathbb{H}$ implies $P = 0$ where "0" means the zero polynomial.*

Proof. We will first show that the assertion is correct for any polynomial of the form

$$s_k(x,y) = \sum_{k_1 i + k_2 j = k} a_{ij} x^i y^j.$$

Note that s_k is not a homogeneous polynomial as the degree may be different for different summands. The summands only satisfy some linear relation! Define

$$\begin{aligned} \mathcal{S} := \{P \mid P \text{ is of the form } s_k \text{ for some } k \in \mathbb{N} \text{ and} \\ P(f,g) \equiv \text{zero on } \overline{\mathbb{H}} \text{ and } P \neq 0\} \end{aligned} \tag{4.1}$$

and assume on the contrary that the assertion was false, i.e. there was a polynomial $P \in \mathcal{S}$ having a structure like s_k for some $k = k(P) \in \mathbb{N}$ then we can also require P to have a minimal k. We will show that there is a Q having a structure of s_l for $l < k$ and still $Q \in \mathcal{S}$. Write P in the form

$$\begin{aligned} P(x,y) &= \sum_{\substack{i,j \\ j=0}} a_{ij} x^i y^j + \sum_{\substack{i,j \\ i\neq 0, j\neq 0}} a_{ij} x^i y^j + \sum_{\substack{i,j \\ i=0}} a_{ij} x^i y^j \\ &= a \cdot x^c + xy \underbrace{\sum_{\substack{i,j \\ i\neq 0, j\neq 0}} a_{ij} x^{i-1} y^{j-1}}_{:=Q(x,y)} + b \cdot y^d \end{aligned} \tag{4.2}$$

where $c = k/k_1, d = k/k_2$, $a \neq 0$ only if $k_1 \mid k$ and $b \neq 0$ only if $k_2 \mid k$. Note that $k(Q) = k(P) - k_1 - k_2 < k(P)$ as $k_1 \neq 0$ and $k_2 \neq 0$. We will show that $a = b = 0$ anyway: into (4.2), we substitute $x = f, y = g$ and obtain a function in z. If we further substitute $z = z_1$, we have

$$0 = P(f,g)(z_1) = a \cdot f(z_1)^c + f(z_1) \underbrace{g(z_1)}_{=0} Q(f,g) + b \cdot \underbrace{g(z_1)^d}_{=0} = a \cdot \underbrace{f(z_1)^c}_{\neq 0}$$

so $a = 0$. Analogously, substituting $z = z_2$ gives $b = 0$. Using continuity arguments and the fact that zeros of holomorphic functions unequal to the zero function do not accumulate, we 'cancel out' f and g and see that $Q(f,g)$ already is the zero function. Thus we arrive at the contradiction. This only works if z_1 and z_2 both are in $\mathbb{H}$ and not in $\mathbb{Q}$. If for example z_1 is a cusp then we may not substitute $z = z_1$ in the above equation. We then proceed as follows:

$$\begin{aligned}
& P(f,g)(z) = 0 \quad \forall z \in \mathbb{H} \\
\Rightarrow \quad & \underbrace{\frac{1}{(L:z)^k}}_{\neq 0 \forall z \in \mathbb{H}} P(f,g)(z) = 0 \quad \forall z \in \mathbb{H} \\
\Rightarrow 0 = & \, a\left(\frac{f(z)}{(L:z)^{k_1}}\right)^c + \sum_{\substack{i,j \\ i\neq 0, j\neq 0}} a_{ij}\left(\frac{f(z)}{(L:z)^{k_1}}\right)^i \left(\frac{g(z)}{(L:z)^{k_2}}\right)^j \\
& \qquad + b\left(\frac{g(z)}{(L:z)^{k_2}}\right)^d \quad (\forall z \in \mathbb{H}) \\
\Rightarrow 0 = & \, a \cdot f_L(z)^c + f_L(z) g_L(z) Q(f_L, g_L)(z) + b \cdot g_L(z)^d \\
\Rightarrow 0 = & \lim_{z \to i\infty} a \cdot f_L(z)^c + f_L(z) g_L(z) Q(f_L, g_L)(z) + b \cdot g_L(z)^d \\
= & \, a \cdot f_L(i\infty)^c + f_L(i\infty) \underbrace{g_L(i\infty)}_{=0} Q(f_L, g_L)(i\infty) + b \cdot \underbrace{g_L(i\infty)^d}_{=0} \\
= & \, a \cdot \underbrace{f_L(i\infty)^c}_{\neq 0}
\end{aligned}$$

Hence, $a = 0$ also follows in this case and analogously $b = 0$.

Note that

$$s_k(f,g) \equiv \text{zero} \Rightarrow s_k = 0$$

is the assertion that we will actually need afterwards. The following general case is just for completeness: Let

$$P(x,y) = \sum_{\substack{i=1,\ldots,n \\ j=1,\ldots,m}} a_{ij} x^i y^j$$

be a polynomial such that $P(f,g) \equiv$ zero i.e. $P(f,g)(z) = 0$ for all $z \in \mathbb{H}$. We first rewrite the polynomial as

$$P(x,y) = \underbrace{\sum_{k_1 i + k_2 j = 1} a_{ij} x^i y^j}_{s_1(x,y)} + \underbrace{\sum_{k_1 i + k_2 j = 2} a_{ij} x^i y^j}_{s_2(x,y)} + \ldots$$

As the sum over the sets $M_1(\Gamma), M_2(\Gamma), \ldots$ is direct and $s_k(f,g) \in M_k(\Gamma)$, $P(f,g) \equiv$ zero implies that for every summand,

$$s_k(f,g) \equiv \text{zero}$$

must hold. Hence, by the first step, $s_k = 0$ for all k and thus $P = 0 + 0 + \ldots + 0 = 0$. $\square$

4.1 Preparatory parameters

Recall from basic number theory Euler's totient function

$$\phi : \mathbb{N} \mapsto \mathbb{N}, \quad \phi(n) = |(\mathbb{Z}/n\mathbb{Z})^*|$$

where the asterisk means "the set of units in $\mathbb{Z}_n$". One can show the following result:

4.1.1 Theorem. *Let $N \in \mathbb{N}$ and $p_1^{e_1} \cdot ... \cdot p_k^{e_k}$ its decomposition into prime factors then the indices of the* inhomogeneous *(!) variants can be computed to be*

$$[SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{j=1}^{k} \left(\frac{1}{p_j} + 1 \right)$$
$$[SL_2(\mathbb{Z}) : \Gamma_1(N)] = \phi(N) \cdot [SL_2(\mathbb{Z}) : \Gamma_0(N)]$$

In particular

$$[SL_2(\mathbb{Z}) : \Gamma_0(2)] = 3, \quad [SL_2(\mathbb{Z}) : \Gamma_1(3)] = 8$$

and since $-Id \in \Gamma_0(2)$ but $-Id \notin \Gamma_1(3)$, we have by Lemma 2.1.8

$$[\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma_0(2)}] = 3, \quad [\widehat{SL_2(\mathbb{Z})} : \widehat{\Gamma_1(3)}] = 4$$

Furthermore one can show that

$$\mathrm{cu}(\Gamma_0(2)) = \{[\![0]\!], [\![\infty]\!]\} = \mathrm{cu}(\Gamma_1(3))$$

Proof. See [Ga 07], section 2.5 and for a sketchy proof for the amount of cusps see [Sko 92]. □

4.2 The case $\Gamma_0(2)$

The following ingredient is the last one needed for simplifying the $k/12-$ Formula for $\Gamma_0(2)$:

4.2.1 Theorem. *The subgroup $\Gamma_0(2)$ possesses only one fixed point orbit. More precisely we have*

$$\mathbb{E}_2(\Gamma_0(2)) = \left\{ \left[\!\!\left[W = \frac{1}{2}(1+i) \right]\!\!\right] \right\} \text{ and } \mathbb{E}_3(\Gamma_0(2)) = \{\}$$

Proof. Concerning $\mathbb{E}_3(\Gamma_0(2))$. Assume that there is a point $z \in \mathbb{H}$ such that $z \in \mathbb{E}_3(\Gamma_0(2))$ i.e. $z = L.\varrho$ and $\Phi(LPL^{-1}) \in \widehat{\Gamma_0(2)}$ so that either $+LPL^{-1}$ or $-LPL^{-1}$ are in $\Gamma_0(2)$. We now show that both cases are impossible. For if $L = \left(\begin{smallmatrix} l_1 & l_2 \\ l_3 & l_4 \end{smallmatrix}\right)$ then $L^{-1} = \left(\begin{smallmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{smallmatrix}\right)$ and therefore

$$\begin{aligned} \pm LPL^{-1} &= \pm \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix} \\ &= \pm \begin{pmatrix} * & * \\ l_3^2 + l_1^2 + l_1 l_3 & * \end{pmatrix} \\ &\equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 2 \qquad \qquad (\text{as } \pm LPL^{-1} \in \Gamma_0(2)) \end{aligned}$$

and the $\pm$-sign is gone as "$+ = -$" in $\mathbb{Z}_2$. A direct case-by-case analysis shows that

$$l_3^2 + l_1^2 + l_1 l_3 \equiv l_3 + l_1 + l_1 l_3 \equiv 0 \mod 2$$

is only possible if $l_1 \equiv l_3 \equiv 0 \mod 2$ so $2 \mid l_1$, $2 \mid l_3$ and thus $2 \mid \det(L) = 1$, a contradiction.

Concerning $\mathbb{E}_2(\Gamma_0(2))$. We first show that $|\mathbb{E}_2(\Gamma_0(2))| = 1$. Let $z = L.i$, $z' = L'.i$ both be representatives of orbits $\mathbb{E}_2(\Gamma_0(2))$, then we have to find a matrix $T \in \Gamma_0(2)$ such that $T.z = z'$. The candidate is clearly $L'L^{-1}$ as

$$L'L^{-1}.z = L'\cancel{L^{-1}L}.i = L'.i = z'.$$

Since $z, z' \in \mathbb{E}_2(\Gamma_0(2))$, $\Phi(LVL^{-1}), \Phi(L'V(L')^{-1}) \in \widehat{\Gamma_0(2)}$. Therefore

$$\begin{aligned} \pm LVL^{-1} &= \pm \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix} \\ &= \pm \begin{pmatrix} * & * \\ l_4^2 + l_3^2 & * \end{pmatrix} \\ &\equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod 2 \end{aligned}$$

so

$$l_4 + l_3 \equiv l_4^2 + l_3^2 \equiv 0 \mod 2 \Longrightarrow l_4 \equiv l_3 \mod 2.$$

and analogously $l'_4 \equiv l'_3 \mod 2$. Hence,

$$L'L^{-1} = \begin{pmatrix} l'_1 & l'_2 \\ l'_3 & l'_4 \end{pmatrix} \begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix}$$
$$\equiv \begin{pmatrix} * & * \\ \underbrace{l'_3}_{\equiv l'_4} l_4 - l'_4 \underbrace{l_3}_{\equiv l_4} & \end{pmatrix} \qquad \equiv \begin{pmatrix} * & * \\ \cancel{l'_4 l_4 - l'_4 l_4} & * \end{pmatrix}$$

and therefore $L'L^{-1} \in \Gamma_0(2)$. It remains to show that W really is a fixed point. A direct calculation shows that

$$T = \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} (-V) \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

has W as a fixed point and lies in $\Gamma_0(2)$. □

Before we simplify the $k/12-$Formula and actually make use of it we clarify that all cusps are regular. This is easily checked: ∞ is regular as $n_{Id} = \widehat{n_{Id}} = 1$ and $0 = V.\infty = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right).\infty$ is regular as $V, (-V) \notin \Gamma_0(2)$ but $\left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right) = V^2 \equiv (-V^2) \mod 2$ are both contained in $\Gamma_0(2)$ so $n_V = \widehat{n_V} = 2$. This assures that the sum over the orders at the cusps only contains integer summands.

With this, $[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma_0(2)}] = 3$ (see thm 4.1.1) and the proof that there is only one fixed point (orbit) modulo $\Gamma_0(2)$ we simplify the $k/12-$Formula for $\Gamma_0(2)$:

$$\omega_f(\infty) + \omega_f(0) + \frac{1}{2}\omega_f(W) + \sum_{\zeta \in \mathcal{J}} \omega_f(\zeta) = \frac{k}{4}$$

There exists no modular form f of negative weight by the dimension formula. One more useful consequence is that f cannot have odd weight neither for if f had odd weight, the left hand side would have a non-integer-part of $1/2$ while the right hand side would have $1/4$, i.e. a direct computation shows that for naturals m, n, k we can never have the relation $m+n/2 = (2k+1)/4$. We have therefore already shown that

$$M_*(\Gamma_0(2)) = \bigoplus_{\substack{k \in \mathbb{N} \\ k \text{ even}}} M_k(\Gamma_0(2))$$

.

Substituting the concrete existing modular forms φ and Θ_2 from Theorems 2.5.7, 2.5.9 into the simplified $k/12-$Formula, we obtain:

$$\underbrace{\omega_\varphi(\infty)}_{\geq 1}+\omega_\varphi(0)+\frac{1}{2}\omega_\varphi(W)+\sum_{\zeta\in\mathcal{J}}\omega_\varphi(\zeta)=\frac{4}{4}=1$$

Since all terms are nonnegative, we can actually conclude that

$$\omega_\varphi(\infty)=1, \quad \omega_\varphi(W)=0,$$

otherwise the left hand side would be strictly bigger than 1 which is impossible.

Finally this gives us the first part of the puzzle we desire for:

$$\varphi(\infty)=0, \quad \varphi(W)\neq 0, \tag{4.3}$$

where $\varphi(\infty)=\lim_{z\to i\infty}\varphi(z)$.

For Θ_2 we obtain

$$\omega_{\Theta_2}(\infty)+\omega_{\Theta_2}(0)+\frac{1}{2}\omega_{\Theta_2}(W)+\sum_{\zeta\in\mathcal{J}}\omega_{\Theta_2}(\zeta)=\frac{2}{4}=\frac{1}{2}$$

Since all terms are nonnegative, we can conclude that

$$\omega_{\Theta_2}(\infty)=0, \quad \omega_{\Theta_2}(W)=1,$$

otherwise the left hand side would be strictly bigger than $1/2$ which is impossible. Again, the $k/12-$ Formula reveals the zeros of the modular form:

$$\Theta_2(\infty)\neq 0, \quad \Theta_2(W)=0. \tag{4.4}$$

Using Theorem 4.0.3 on φ and Θ_2 which behave differently at two points according to (4.3), (4.4), (note that the term "different behavior" in this theorem also includes "different behavior at the cusps") we now know that φ, Θ_2 are algebraically independent. With this, we can realize step ③, "we can construct these x essentially different modular forms" from the introduction .'Essentially different' here means linearly independent. Since odd weights cannot occur, let k be a given even natural weight, then

$$\Theta_2^{(k/2)}\varphi^0, \Theta_2^{(k/2)-2}\varphi^1, \Theta_2^{(k/2)-4}\varphi^2, ..., \Theta_2^{(k/2)-2r}\varphi^r, ...$$

are linearly independent modular forms of weight $[(k/2)-2r]\cdot 2+r\cdot 4=k-4r+4r=k$. We can do this as long as

$$\frac{k}{2}-2r\geq 0 \iff \frac{k}{4}\geq r \iff \left\lfloor\frac{k}{4}\right\rfloor+1\geq r \quad (\text{since } r\in\mathbb{N})$$

So we find $\lfloor k/4 \rfloor + 1$ linearly independent modular forms. Now step ② from the introduction comes into play: the dimension formula implies that

$$\dim(M_k(\Gamma_0(2))) \leq \left\lfloor k[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma_0(2)}]/4 \right\rfloor + 1 = \lfloor k/4 \rfloor + 1$$

Consequently, we have already found all modular forms of weight k and therefore have shown

4.2.2 Theorem. *Set* $r = \lfloor k/4 \rfloor + 1$, *then*

$$M_k(\Gamma_0(2)) = \mathrm{Lin}_{\mathbb{C}}(\Theta_2^{k/2}, \Theta_2^{k/2-2}\varphi, ..., \Theta_2^{k/2-2r}\varphi^r) := N_k(\Gamma_0(2))$$

and therefore

$$M_*(\Gamma_0(2)) = \bigoplus_{\substack{k\in\mathbb{N} \\ k \ even}} M_k(\Gamma_0(2)) = \bigoplus_{\substack{k\in\mathbb{N} \\ k \ even}} N_k(\Gamma_0(2)) = \mathbb{C}[\Theta_2, \varphi]$$

4.3 The case $\Gamma_1(3)$

Analogously to the case of $\Gamma = \Gamma_0(2)$ we need some more knowledge about the fixed points:

4.3.1 Theorem. *The subgroup* $\Gamma_1(3)$ *possesses only one fixed point orbit. More precisely we have*

$$\mathbb{E}_2(\Gamma_1(3)) = \{\}, \text{ and } \mathbb{E}_3(\Gamma_1(3)) = \left\{ \left[\!\left[Q = \frac{1}{6}(3+\sqrt{3}i) \right]\!\right] \right\}$$

Proof. Concerning $\mathbb{E}_2(\Gamma_1(3))$. Assume that there is a point $z \in \mathbb{H}$ such that $[\![z]\!] \in \mathbb{E}_2(\Gamma_1(3))$ i.e. $z = L.\varrho$ and $\Phi(LVL^{-1}) \in \widehat{\Gamma_1(3)}$ so that either $+LVL^{-1}$ or $-LVL^{-1}$ are in $\Gamma_1(3)$. We now show that both cases are impossible. For if $L = \begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}$ then $L^{-1} = \begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix}$ and therefore

$$\begin{aligned} +LVL^{-1} &= +\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix} \\ &= +\begin{pmatrix} l_2l_4 + l_1l_3 & * \\ l_4^2 + l_3^2 & -l_2l_4 - l_3l_1 \end{pmatrix} \\ &\equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \qquad\qquad (\text{as } LVL^{-1} \in \Gamma_1(3)) \end{aligned}$$

For $x, y \in \mathbb{Z}_3$ we have the following possibilities (all values modulo 3):

Table 2: Computations in $\mathbb{Z}_3$

x	y	x^2	y^2	xy	x^2+y^2	x^2+y^2+xy
0	0	0	0	0	0	0
0	1	0	1	0	1	1
0	2	0	1	0	1	1
1	0	1	0	0	1	1
1	1	1	1	1	2	0
1	2	1	1	2	2	1
2	0	1	0	0	1	1
2	1	1	1	2	2	1
2	2	1	1	1	2	0

We see that

$$l_4^2 + l_3^2 \equiv 0 \mod 3 \Longrightarrow l_3 \equiv l_4 \equiv 0 \mod 3$$

and therefore $3|\det(L) = 1$, a contradiction. In the case $-LVL^{-1} \in \widehat{\Gamma_1(3)}$ nothing changes because

$$l_4^2 + l_3^2 \equiv 0 \mod 3 \iff (-1)(l_4^2 + l_3^2) \equiv 0 \mod 3$$

Concerning $\mathbb{E}_3(\Gamma_1(3))$. We first show that $|\mathbb{E}_3(\Gamma_1(3))| = 1$. Let $z = L.i,\ z' = L'.i$ both be representatives of orbits in $\mathbb{E}_3(\Gamma_1(3))$, then we show again that $L'L^{-1} \in \Gamma_1(3)$. Since $[\![z]\!], [\![z']\!] \in \mathbb{E}_3(\Gamma_1(3))$, $\Phi(LPL^{-1}), \Phi(L'P(L')^{-1}) \in \widehat{\Gamma_1(3)}$. It cannot be the case that $+LPL^{-1} \in \Gamma_1(3)$ as

$$\begin{aligned} &+LPL^{-1} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \\ &\Longrightarrow 1 = \operatorname{tr}(P) = \operatorname{tr}(LPL^{-1}) \equiv \operatorname{tr}\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \equiv 2 \mod 3 \end{aligned}$$

This means $-LPL^{-1} \in \Gamma_1(3)$ and analogously $-L'P(L')^{-1} \in \Gamma_1(3)$, so

$$\begin{aligned} -LPL^{-1} &= -\begin{pmatrix} l_1 & l_2 \\ l_3 & l_4 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_1 \end{pmatrix} \\ &= \begin{pmatrix} -l_1l_3 + l_2l_3 - l_2l_4 & * \\ (-1)(l_4^2 + l_3^2 + l_3l_4) & l_1l_3 - l_1l_4 + l_2l_4 \end{pmatrix} \\ &\equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod 3 \end{aligned}$$

We have $l_3 \equiv l_4 \equiv x \mod 3$ (compare the last column of Table 2). Substituting this into the diagonal entries we obtain

$$-l_1x + \cancel{l_2x} - \cancel{l_2x} \equiv 1 \mod 3, \quad \cancel{l_1x} - \cancel{l_1x} + l_2x \equiv 1 \mod 3. \tag{4.5}$$

The case $x \equiv 0$ cannot occur as otherwise $3 \mid l_3, l_4 \Rightarrow 3 \mid \det(L) = 1$. Also note that due to (4.5), the values of l_1 and l_2 (modulo 3) are uniquely determined once x is determined. Completely analogously one shows that $0 \not\equiv x' \equiv l'_3 \equiv l'_4$ and equation (4.5) holds for L' too.
With

$$\begin{aligned} L'L^{-1} &= \begin{pmatrix} l'_1 & l'_2 \\ l'_3 & l'_4 \end{pmatrix} \begin{pmatrix} l_4 & -l_2 \\ -l_3 & l_4 \end{pmatrix} \\ &= \begin{pmatrix} l_4l'_1 - l_3l'_2 & * \\ l_4l'_3 - l_3l'_4 & l_1l'_4 - l_2l'_3 \end{pmatrix} \\ &\equiv \begin{pmatrix} x(l'_1 - l'_2) & * \\ xx' - xx' \equiv 0 & x'(l_1 - l_2) \end{pmatrix} \end{aligned}$$

we see that in all four cases either $+L'L^{-1} \in \Gamma_1(3)$ or $-L'L^{-1} \in \Gamma_1(3)$ (compare table 4.3), and since $\pm L'L^{-1}z = z'$ we have $|\mathbb{E}_3(\Gamma_1(3))| = 1$.

Table 3: Cases for x, x' and resulting matrix $L'L_{-1}$ (mod 3)

x	x'	l_1	l_2	l_3	l_4	l'_1	l'_2	l'_3	l'_4	$L'L^{-1}$
1	1	-1	1	1	1	-1	1	1	1	$\begin{pmatrix} -2 & * \\ 0 & -2 \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$
1	2	-1	1	1	1	1	2	2	2	$\begin{pmatrix} (1-2) & * \\ 0 & 2(-2) \end{pmatrix} \equiv \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$
2	1	1	2	2	2	-1	1	1	1	$\begin{pmatrix} 2(-2) & * \\ 0 & (1-2) \end{pmatrix} \equiv \begin{pmatrix} -1 & * \\ 0 & -1 \end{pmatrix}$
2	2	1	2	2	2	1	2	2	2	$\begin{pmatrix} -2 & * \\ 0 & -2 \end{pmatrix} \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$

It remains to show that Q really is a fixed point. A direct calculation shows that

$$T = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} (-P) \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$$

has Q as a fixed point and lies in $\Gamma_1(3)$. □

After verifying that all cusps are regular, using $[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma_1(3)}] = 4$, $\mathrm{cu}(\Gamma_1(3)) = \{[\![\infty]\!], [\![0]\!]\}$ (see thm 4.1.1) and the result above we can simplify the $k/12$ – Formula for $\Gamma_1(3)$:

$$\omega_f(\infty) + \omega_f(0) + \frac{1}{3}\omega_f(Q) + \sum_{\zeta \in \mathcal{J}} \omega_f(\zeta) = \frac{k}{3}$$

Note that although the index $[\mathrm{SL}_2(\mathbb{Z}) : \Gamma_1(3)]$ is 8, for the $k/12-$Formula we need the index $[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma_1(3)}]$ which is $4 = 8/2$ due to Lemma 2.1.8 because $-Id \notin \Gamma_1(3)$.

We now proceed completely analogously to the case of $\Gamma_0(2)$. Again, no modular form with negative weight exists, i.e.

$$M_*(\Gamma_1(3)) = \bigoplus_{k\in\mathbb{N}} M_k(\Gamma_1(3))$$

.

Substituting the concrete existing modular forms ψ and Θ_1 from Theorems 2.5.7, 2.5.9 into the simplified $k/12-$Formula, we obtain:

$$\underbrace{\omega_\psi(\infty)}_{\geq 1} + \omega_\psi(0) + \frac{1}{3}\omega_\psi(Q) + \sum_{\zeta\in\mathcal{J}} \omega_\psi(\zeta) = \frac{3}{3} = 1$$

Since all terms are nonnegative, we can conclude that

$$\omega_\psi(\infty) = 1, \quad \omega_\psi(Q) = 0,$$

otherwise the left hand side would be strictly bigger than 1 which is impossible. This directly implies

$$\psi(\infty) = 0, \quad \psi(Q) \neq 0, \tag{4.6}$$

where $\psi(\infty) = \lim_{z\to i\infty} \psi(z)$.

For Θ_1 we obtain

$$\omega_{\Theta_1}(\infty) + \omega_{\Theta_1}(0) + \frac{1}{3}\omega_{\Theta_1}(Q) + \sum_{\zeta\in\mathcal{J}} \omega_{\Theta_1}(\zeta) = \frac{1}{3}$$

Since all terms are nonnegative, we can conclude that

$$\omega_{\Theta_1}(\infty) = 0, \quad \omega_{\Theta_1}(Q) = 1,$$

otherwise the left hand side would be strictly bigger than 1/3 which is impossible. Again, the $k/12-$Formula reveals the zeros of the modular form:

$$\Theta_1(\infty) \neq 0, \quad \Theta_1(Q) = 0. \tag{4.7}$$

Using Theorem 4.0.3 on ψ and Θ_1 which behave differently at two points according to (4.6) and (4.7), we now know that ψ, Θ_1 are algebraically independent. Let k be a given natural weight, then

$$\Theta_1^k\psi^0, \Theta_1^{k-3}\varphi^1, \Theta_1^{k-6}\psi^2, ..., \Theta_1^{k-3r}\varphi^r, ...$$

are linearly independent modular forms of weight $(k-3r)\cdot 1 + r\cdot 3 = k-3r+3r = k$. We can do this as long as

$$k-3r \geq 0 \iff \frac{k}{3} \geq r \iff \left\lfloor \frac{k}{3} \right\rfloor + 1 \geq r \quad (\text{since } r \in \mathbb{N})$$

So we find $\lfloor k/3 \rfloor + 1$ linearly independent modular forms. The dimension formula implies that

$$\dim(M_k(\Gamma_1(3))) \leq \left\lfloor k[\widehat{\mathrm{SL}_2(\mathbb{Z})} : \widehat{\Gamma_1(3)}]/12 \right\rfloor + 1 = \lfloor k/3 \rfloor + 1$$

Consequently, we have already found all modular forms of weight k and therefore have shown

4.3.2 Theorem. *Set* $r = \lfloor k/3 \rfloor + 1$*, then*

$$M_k(\Gamma_1(3)) = \mathrm{Lin}_{\mathbb{C}}(\Theta_1^k, \Theta_1^{k-3}\psi, ..., \Theta_1^{k-3r}\psi^r) := N_k(\Gamma_1(3))$$

and therefore

$$M_*(\Gamma_1(3)) = \bigoplus_{k \in \mathbb{N}} M_k(\Gamma_1(3)) = \bigoplus_{k \in \mathbb{N}} N_k(\Gamma_1(3)) = \mathbb{C}[\Theta_1, \psi]$$

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